Waveguide Analog of Tunneling through Quantum Potential Barriers*

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Expressions are derived for the electromagnetic fields of guided waves which are analogous to the quantum-mechanical equations representing barrier tunneling. This analogy is achieved by comparing the propagation constant of the Schrödinger time-independent wave equation with that of the electromagnetic wave equation in waveguide and by comparing the de Broglie wavelength of a particle with the wavelength of the waves that propagate the energy. This results in an expression relating the form of an arbitrary one-dimensional energy barrier to the physical dimensions of a section of waveguide. The analogy is tested by the propagation of energy in the TE_{10} mode at both the 3- and 6-cm bands for the cases of rectangular and hyperbolic barriers. Although evanescent modes are present at the discontinuous regions, the analog for the rectangular barrier, which is considered to be the worse case, is verified when an effective barrier length l_{eff} of about $1.2l$ is used. This experimental verification demonstrates the possibility of waveguide simulation of quantum-mechanical energy barriers and the practicality of utilizing an electromagnetic analog for demonstrating the tunneling phenomenon and provides a method for measurement of the transmission coefficient through an arbitrarily shaped barrier.

INTRODUCTION

Wave propagation, whether electromagnetic, mechanical, or acoustic, is usually described in terms of solutions to wave equations. Analogies in which one type of propagation is described in terms of another become very useful, for they may provide solutions to problems that might otherwise be difficult to solve. Analog relations have been shown to exist[1] between wave propagation in crystal structures and electromagnetic wave propagation down a transmission line. Periodic structures[2] in waveguides have the pass- and stop-band properties characteristic of crystal structure. In 1924 de Broglie ascribed a wavelength λ to quantum particles where $\lambda = h/p$, h being Planck's constant and p, the linear momentum of the particle. Experiments verifying the wave properties of quantum particles have been performed many times.

Since the wavefunction that describes quantum barrier penetration by particles decays exponentially, there is reason to believe that this phenomenon may be expressed classically in terms of guided (acoustic, electric, mechanical, etc.) wave propagation. The subject matter reported, therefore, is the investigation of a classical analog to particle tunneling through quantum potential barriers, utilizing electromagnetic propagation through a section of below-cutoff waveguide.

WAVEGUIDES

The most commonly used waveguide is the rectangular waveguide. The propagation of energy in this guide is represented as normal mode solutions to the wave equation. By suitable choice of waveguide dimensions, all modes higher than the dominant mode are eliminated. Because of the mathematical simplicity and apparatus vailable, the TE_{10} mode is selected for the study.

The TE_{10} wave is a plane electromagnetic wave whose **E** and **H** vectors are oriented so that the components $E_x = H_y = E_z = 0$. The longitudinal component satisfies the time-independent equation $(d^2/dz^2)H_z + k^2H_z = 0$ where $k^2 = \omega^2\mu e - (\pi/a)^2$, a is the guide cross-section length in the x direction, $H_z = \cos(\pi x/a)e^{ikz}$, and by Maxwell's

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[1] L. Brillouin, *Wave Propagation in Periodic Structures* (Dover Publications, Inc., New York, 1953).

[2] R. E. Collin, *Field Theory of Guided Waves* (McGraw-Hill Book Company, New York, 1960), Chap. 9.

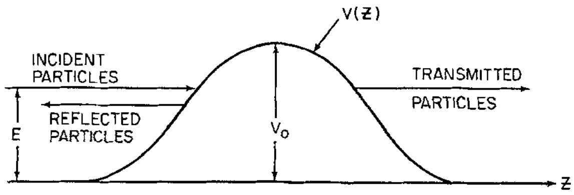


FIG. 1. Particles incident upon quantum potential barrier have finite probability of transmission.

equation,

$$E_y = A \sin(\pi x/a) e^{ikz},$$

$$H_x = -Ak/\omega\mu \sin(\pi x/a) e^{ikz},$$

where $A = i\omega\mu a/\pi$.

QUANTUM TUNNELING

Quantum tunneling phenomena have their origin in the wave nature of particles as described by the solutions to the time-independent Schrödinger equation $\nabla^2\psi(r) + k^2\psi(r) = 0$ where $\psi(r)$ is the Schrödinger wavefunction,

$$k^2 = (2m/\hbar^2)[E - V(r)],$$

E is the particle energy, m is particle mass, $\hbar = h/2\pi$, and the potential $V(r)$ may have any functional form.

Let us consider the one-dimensional problem of particle transmission through a region having a potential energy greater than the particle energy, as seen in Fig. 1. Classically the particle rebounds. According to quantum mechanics, there is a probability of penetration through the region. The transmission coefficient is a measure of this probability. The analysis involves finding solutions to the time-independent equation $(d^2/dz^2)\psi + k^2\psi = 0$ where $k^2 = (2m/\hbar^2)[E - V(z)]$.

The form of this equation suggests an analogy between the waveguide and quantum-mechanical propagation constants. This is somewhat unconventional in the sense that the usual electrical-mechanical analog uses electronic elements (inductors, capacitors, etc.) to describe mechanical phenomena. Moreover, the propagation constant in waveguide is a function of the geometrical structure and therefore offers a physical insight into the mathematical description of the quantum barrier. With this in mind, the guide properties are analyzed geometrically in Table I and Fig. 2.

TABLE I. Comparison of quantum barrier potential and waveguide discontinuity regions.

In region I of Fig. 2(a)	In region I of Fig. 2(b)
$k = (2mE/\hbar^2)^{\frac{1}{2}} = 2\pi/\lambda_E$	$k = [\omega^2\mu\epsilon - (\pi/a)^2]^{\frac{1}{2}}$
where λ_E = deBroglie wavelength of particle.	where $2a = \lambda_c$, the cutoff wavelength f is generator freq. λ is wavelength of freq. $k = (2\pi/\lambda)[1 - (\lambda/2a)^2]^{\frac{1}{2}}$
In region II of Fig. 2(a)	**In region II of Fig. 2(b)**
if $E > V_0$	if $\lambda < 2a'$
$k = \{(2m/\hbar^2)[E - V(z)]\}^{\frac{1}{2}}$ $= (2\pi/\lambda_E)[1 - V(z)E]^{\frac{1}{2}}$	$k = (2\pi/\lambda)\{1 - [\lambda/2a(z)]^2\}^{\frac{1}{2}}$ if $\lambda > 2a'$
if $E < V_0$	
$k = \{(2m/\hbar^2)[V(z) - E]\}^{\frac{1}{2}}$ $= (2\pi/\lambda_E)[V(z)/E - 1]^{\frac{1}{2}}$.	$k = (2\pi/\lambda)\{[\lambda/2a(z)]^2 - 1\}^{\frac{1}{2}}$

Let s be a proportionality constant between the propagation constants of the two systems. Then

$$k(\text{QM}) = sk \text{ (waveguide)}, \tag{1}$$

$$2\pi/\lambda_E = (2\pi s/\lambda)[1 - (\lambda/2a)^2]^{\frac{1}{2}} \quad \text{for } V(z) = 0,\ \lambda < 2a, \tag{2a}$$

$$(2\pi/\lambda_E)\{1 - [V(z)/E]\}^{\frac{1}{2}} = (2\pi s/\lambda)\{1 - [\lambda/2a(z)]^2\}^{\frac{1}{2}} \quad \text{for } E > V_0,\ \lambda < 2a, \tag{2b}$$

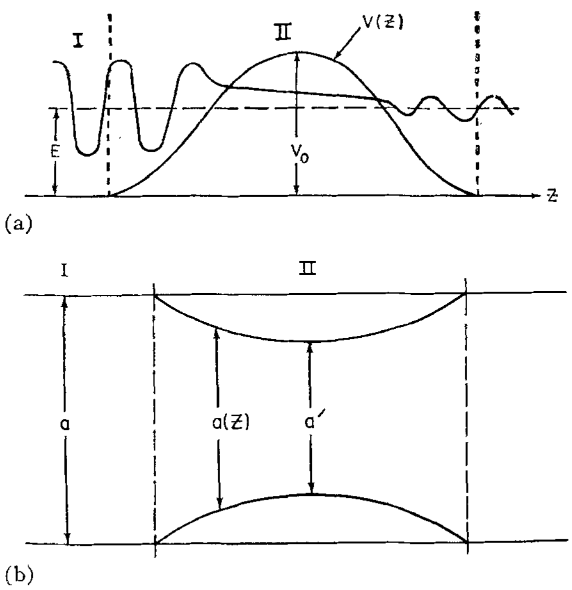


FIG. 2. Comparison of (a) quantum barrier potential and (b) waveguide discontinuous regions.

$$(2\pi/\lambda_E)\{[V(z)/E]-1\}^{\frac{1}{2}} = (2\pi s/\lambda)\{[\lambda/2a(z)]^2-1\}^{\frac{1}{2}} \quad \text{for } E<V_0,\ 2a'<\lambda<2a. \quad (2c)$$

Let us consider the case for $E<V_0$. Dividing (2*a*) by (2*c*) and squaring, we have

$$\frac{E}{V(z)}=\frac{(2a/\lambda)^2-1}{[a/a(z)]^2-1}. \quad (3)$$

If we assume that the generator frequency simulates the particle kinetic energy, then as $E\rightarrow V_0$, $\lambda\rightarrow 2a'$ and

$$\frac{V_0}{V(z)}=\frac{(a/a')^2-1}{[a/a(z)]^2-1}.$$

In general $V(z)=V_0 g(z)$, where $g(z)$ is the barrier function, and V_0 is the maximum value of the positively shaped barrier or is the minimum value for the negatively shaped barrier. Then

$$g(z)=\frac{[a/a(z)]^2-1}{(a/a')2-1}$$

and

$$a(z)=a\{1+g(z)[(a/a')^2-1]\}^{\frac{1}{2}}. \quad (4)$$

To describe this function in terms of the coordinate system, we note that

$$x(z)=\tfrac{1}{2}[a-a(z)]. \quad (5)$$

RECTANGULAR BARRIER

The rectangular barrier (see Fig. 3) has one of the simplest potential shapes used in the analysis of barrier penetration $V(z)=V_0 g(z)$ where $g(z)=0$ for $z\leq 0$ and $z\geq l_v$ and $g(z)=1$ for $0<z<l_v$. The transmission coefficient[3] for this barrier is

$$T^{-1}=1+\frac{1}{4}\frac{\sinh^2 k'l_v}{(E/V_0)(1-E/V_0)} \quad \text{for } E<V_0, \quad (6a)$$

where

$$k'=(2\pi/\lambda_V)(1-E/V_0)^{\frac{1}{2}}$$

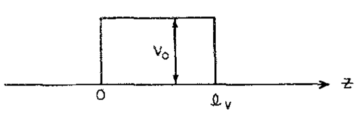


FIG. 3. Rectangular barrier.

[3] R. N. Eisberg, *Fundamentals of Modern Physics* (John Wiley & Sons, Inc., New York, 1961), pp. 233–234.

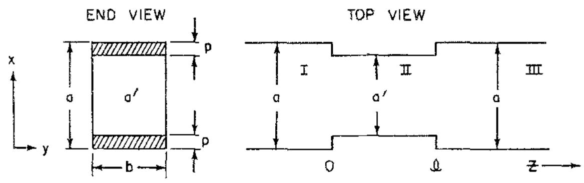


FIG. 4. Waveguide design for simulation of rectangular potential barriers.

and

$$T^{-1}=1+\frac{1}{4}\frac{\sin^2 k'l_v}{(E/V_0)(E/V_0-1)} \quad \text{for } E>V_0, \quad (6b)$$

where $k'=(2\pi/\lambda_V)(E/V_0-1)^{\frac{1}{2}}$ and λ_V is the deBroglie wavelength equal to that particle wavelength whose kinetic energy equals the barrier energy. From (2a) we find that as $\lambda_E\rightarrow\lambda_V$, $\lambda\rightarrow 2a$; therefore

$$1/\lambda_V=(S/a')[1-(a'/a)^2]^{\frac{1}{2}}. \quad (7)$$

If we let q be a proportionality constant between the barrier length l_V and the length of the waveguide discontinuity l, then

$$l_V=ql \quad (8a)$$

and

$$l_V/\lambda_V=(Ml/a')[1-(a'/a)^2]^{\frac{1}{2}}, \quad (8b)$$

where M is the dimensionless constant qs. The waveguide contour for the simulated barrier is shown in Fig. 4.

Discontinuities[4] in waveguides give rise to reflected waves and excite higher-order evanescent modes which decay exponentially with distance. From symmetry considerations, only those modes that are symmetrical about $x=a/2$ are excited by the discontinuity. With a TE_{10} mode incident from the region $z<0$, the electromagnetic field in all three regions is expressed in terms of the transverse components.

Region I:

$$E_y^{(1)}=A_1(e^{ikz}+re^{-ikz})\sin\left(\frac{\pi x}{a}\right)+\sum_{n=3,5,\cdots}^{\infty} A_n\sin\left(\frac{n\pi x}{a}\right)e^{-k_n z},$$

$$H_x^{(1)}=Y_1 A_1(e^{ikz}-re^{-ikz})\sin\left(\frac{\pi x}{a}\right)+\sum_{n=3,5,\cdots}^{\infty} Y_n A_n\sin\left(\frac{n\pi x}{a}\right)e^{-k_n z},$$

[4] R. E. Collin, *Field Theory of Guided Waves* (McGraw–Hill Book Company, New York, 1960), p. 316.

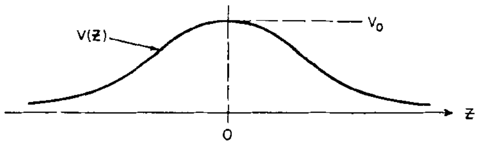


FIG. 5. Hyperbolic barrier.

where

$$k=\omega\mu\epsilon-(\pi/a)^2,\quad k_n=(n\pi/a)^2-\omega^2\mu\epsilon,$$
$$Y_1=k/\omega\mu;\quad Y_n=kn/i\omega\mu,\quad A_1=i\omega\mu a/\pi.$$

Region II:

$$E_y^{(2)}=\sum_{n=1,3,\cdots}^{\infty} B_n \sin\left[\frac{n\pi}{a'}(x-p)\right]e^{k_n z}+\sum_{n=1,3,\cdots}^{\infty} C_n \sin\left[\frac{n\pi}{a'}(x-p)\right]e^{-k_n z},$$

$$H_x^{(2)}=\sum_{n=1,3,\cdots}^{\infty} Y_n'B_n \sin\left[\frac{n\pi}{a'}(x-p)\right]e^{k_n' z}+\sum_{n=1,3,\cdots}^{\infty} Y_n'C_n \sin\left[\frac{n\pi}{a'}(x-p)\right]e^{-k_n z},$$

where $k_n'=(n\pi/a')^2-\omega^2\mu\epsilon$, $Y_n'=kn'/i\omega\mu$.

Region III:

$$E_y^{(3)}=t\sin\left(\frac{\pi x}{a}\right)e^{ikz}+\sum_{n=3,5,\cdots}^{\infty} D_n \sin\left(\frac{n\pi x}{a}\right)e^{k_n z},$$

$$H_x^{(3)}=-Y_1 t\sin\left(\frac{\pi x}{a}\right)e^{ikz}-\sum_{n=3,5,\cdots}^{\infty} Y_n D_n \sin\left(\frac{n\pi x}{a}\right)e^{k_n z}.$$

The transverse fields are continuous in all three regions and are equal at the boundaries. In general, solutions for the transmission amplitude involve integral equations derived by using the orthogonality properties of the field equations. Because of the discontinuous boundary regions, these equations are lengthy and the solutions are approximate. This difficulty is avoided by using a different approach, whereby analysis of only the lowest-order mode is considered. The analysis is justified on the basis that only the dominant mode propagates in the above cutoff region, and that most of the power transferred in the cutoff region is accomplished only by a combination of the lowest-order nonpropagating modes, which decay in opposite directions. Although the analysis leads to approximate solutions, the accuracy of the analog may be improved by a correction factor. Equating the fields at the boundaries and solving for t/A_1, we obtain the power transmission coefficient $T=tt^*/A_1A_1^*$, which is approximately

$$T^{-1}\approx 1+\tfrac{1}{4}[(k'/k)^2+2]\sinh^2 k'l,$$

where

$$k=(2\pi/\lambda)[1-(\lambda/2a)^2]^{\frac{1}{2}}$$

and

$$k'=(2\pi/\lambda)[(\lambda/2a')^2-1]^{\frac{1}{2}}\quad \text{for } 2a'<\lambda<2a.$$

By inserting the conditions

$$\begin{Bmatrix} V(z)=V_0 \\ a(z)=a' \end{Bmatrix}$$

into (3), we have the analog relations

$$(k'/k)^2+(k/k')^2+2=[(E/V_0)(1-E/V_0)]^{-1}.$$

By taking the product of (1) and (8a), we have $k'l_v(\text{QM})=Mk'l$ (waveguide). Except for the constant M in the argument of the hyperbolic function, the waveguide and potential barrier transmission coefficients are identical.

It should be remembered that the waveguide transmission equation is derived from the assumption that only the lowest-order evanescent mode transfers the power. Since higher-order modes do transfer some power as reflections at the discontinuity regions, additional apparent signal attenuation of the dominant mode is to be expected. This additional attenuation may be regarded as the effect due to a longer effective

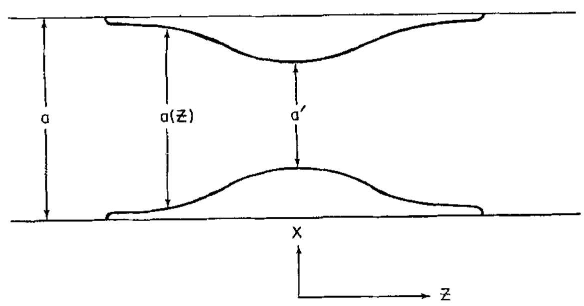


FIG. 6. Waveguide simulation of hyperbolic barrier.

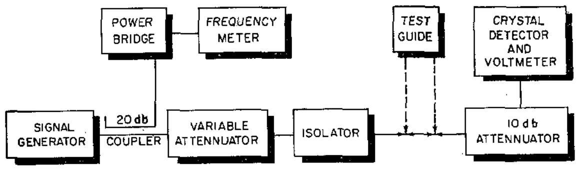


FIG. 7. Block diagram of test setup.

guide length. The value of M therefore is considered as a modification factor that corrects the analogy between the guide and barrier dimensions. Since a slowly varying discontinuous region produces few higher-order modes, attenuation is comparable to that caused by a similarly shaped potential barrier. For an ideal analog, the value of M should be unity.

HYPERBOLIC BARRIER

A slowly varying potential barrier that has the mathematical form $g(z)=\cosh^{-2}\eta z$, where η is an arbitrary constant, is called a hyperbolic barrier. In our case it is convenient to let $\eta=\alpha/M$, where α is arbitrary. The transmission coefficient[5] is

$$T^{-1}=1+\frac{\cosh^2[(\pi/2)(1-8mV_0M^2/\hbar^2\alpha^2)^{\frac{1}{2}}]}{\sinh^2(M\pi k/\alpha)}\quad \text{for } \frac{8mV_0M^2}{\hbar^2\alpha^2}<1,$$

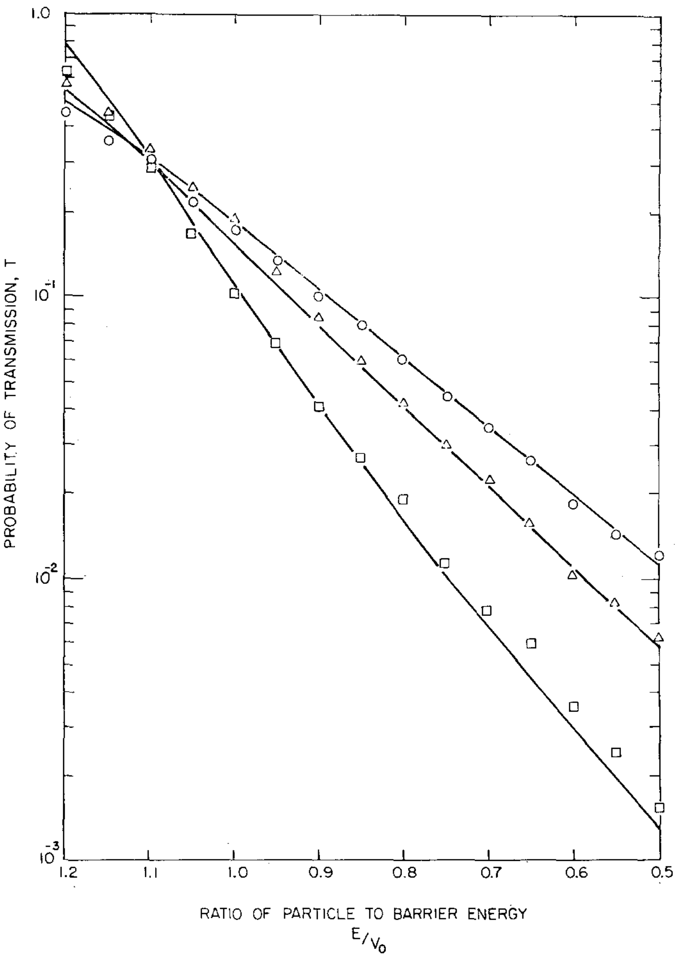
$$T^{-1}=1+\frac{\cosh^2[(\pi/2)(8mV_0M^2/\hbar^2\alpha^2-1)^{\frac{1}{2}}]}{\sinh^2(M\pi k/\alpha)}\quad \text{for } \frac{8mV_0M^2}{\hbar^2\alpha^2}>1,$$

where $k=(2mE/\hbar^2)^{\frac{1}{2}}$.

FIG. 8. Ratio of particle to barrier energy. *Transmission coefficient for rectangular barrier*:

$$T=\left[1+\frac{1}{4}\frac{\sinh^2(2\pi lv/\lambda v)(1-E/V_0)^{\frac{1}{2}}}{(E/V_0)(1-E/V_0)}\right]^{-1}$$

for $E/V_0<1$. *Waveguide experiment*: ○, $lv/\lambda v=0.66$; △, $lv/\lambda v=0.75$; □, $lv/\lambda v=0.90$; —, theoretical quantum-mechanics curves.



[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics Nonrelativistic Theory* (Addison-Wesley Publ. Company, Inc., Reading, Mass., 1958), pp. 76–77.

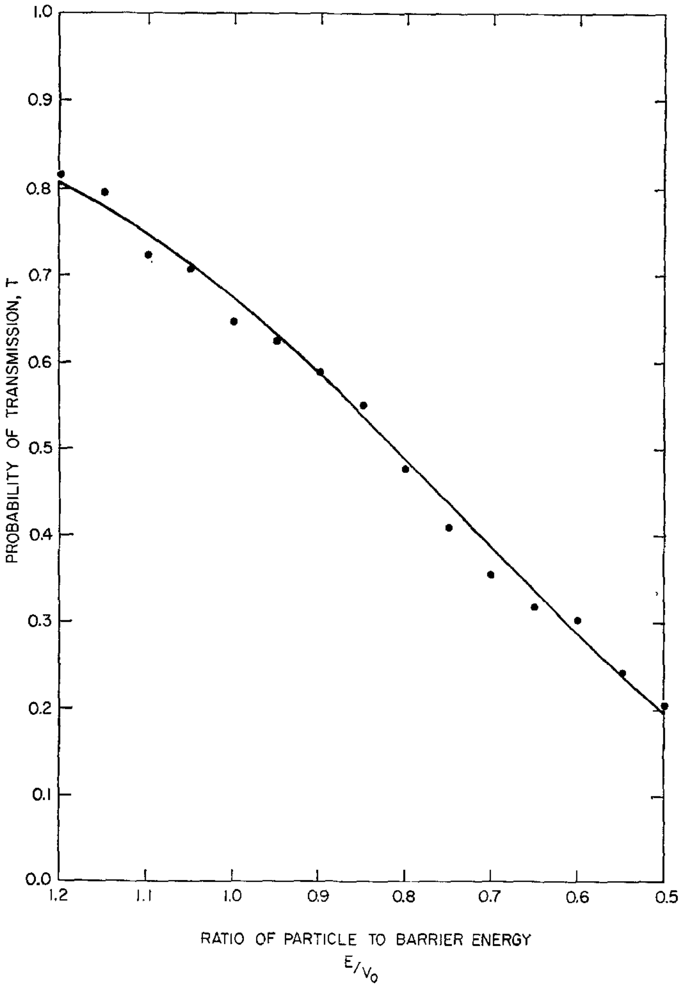


FIG. 9. Ratio of particle to barrier energy. *Transmission coefficient for hyperbolic barrier:*

$$V(Z) = V_0 \cosh^{-2}[(\alpha/m)Z].$$

$$T = \left[1 + \frac{\cosh^2\pi(M^2-0.25)^{\frac{1}{2}}}{\sinh^2 M\pi(E/V_0)^{\frac{1}{2}}}\right]^{-1}.$$

Wavgeuide experiment curve: ●. Quantum-mechanical theoretical curve: —.

The value of α, being arbitrary, is selected to be $\alpha = (2mV_0/\hbar^2)^{\frac{1}{2}}$. Then, $8mV_0/\hbar^2\alpha^2 = 4$, and

$$T^{-1} = 1 + \frac{\cosh^2[(\pi/2)(4M^2-1)^{\frac{1}{2}}}{\sin^2 M\pi(E/V_0)^{\frac{1}{2}}}.$$

The waveguide analog of this barrier is described by (4) as

$$a(z) = a\{1 + [(a/a')^2 - 1] \cosh^{-2}\alpha z/M\}^{-\frac{1}{2}}.$$

From (2a) we see that as $\lambda_E \rightarrow \lambda_{v0}\lambda \rightarrow 2a'$, and therefore

$$\alpha = 2\pi/\lambda_{V_0} = (\pi s/a')[1 - (a'/a)^2]^{\frac{1}{2}}.$$

If we let q be a proportionality constant between the barrier length and the length of the waveguide discontinuous region, then

$$z(\text{quantum barrier}) = qz(\text{waveguide})$$

and

$$(\alpha z/M)(\text{quantum barrier}) = (\pi z/a')[1 - (a'/a)^2]^{\frac{1}{2}}(\text{waveguide})$$

where $M = qS$. The barrier is therefore described in the waveguide by

$$\left[\frac{a}{a(z)}\right]^2 = 1 + \left[\left(\frac{a}{a'}\right)^2 - 1\right] \times \cosh^{-2}\left\{\left(\frac{\pi z}{a'}\right)\left[1 - \left(\frac{a'}{a}\right)^2\right]^{\frac{1}{2}}\right\}.$$

The waveguide contour for the hyperbolic barrier is shown in Fig. 6.

EXPERIMENT

The experimental test setup is that of Fig. 7. The microwave signal generator is amplitude-modulated internally at 1 kc/sec. A power bridge and frequency meter monitor the input signal through a 20-dB directional coupler. Initially the variable attenuator is adjusted to provide linear output crystal drive. Then the test guide is inserted, as shown, and the variable attenuator is adjusted to obtain the same voltmeter deflection (substitution method). The change in attenuator readings is a measure of the transmission coefficient. This procedure is repeated at each frequency setting.

The test waveguide with a rectangular discontinuity is a section of x-band waveguide with dimensions $a=2.286$ cm, $b=1.016$ cm, and with cutoff frequency $f_c=6.557$ Gc/sec. The discontinuous region is centrally located in the guide and has dimensions $a=1.5$ cm, $b=1.016$ cm, $l=3.0$ cm, 2.5 cm, 2.2 cm, and with cutoff frequency $f_c'=10.0$ Gc/sec. From (8b),

$$l_v/\lambda_v=(\lambda/3.0)M[1-(6.557/10.0)^2]^{\frac{1}{2}}$$

$l_v/M\lambda_v=0.755$, 0.629, 0.554. The measured values of transmission coefficient (output power divided by input power, experimental points) as a function of frequency (as expressed in terms of E/V_0) are plotted in Fig. 8. The points follow the theoretical curve for $l_v/\lambda_v=0.9$, 0.75, 0.66, indicating that $M=1.19$ (approx.).

A test waveguide was fabricated to simulate the hyperbolic barrier. The dimensions are $a=4.755$ cm, $a'=2.725$ cm, with cutoff frequencies $f_c=3152$, $f_c'=5500$ Mc/sec. The overall barrier length, as measured from the center, is 3.0 cm, where departure from the actual hyperbolic shape is negligible. The transmission coefficient for the simulated barrier is displayed along with the theoretical quantum mechanic curve (for $M=1.15$) in Fig. 9. Good correlation is obtained over the energy (frequency) range plotted.

CONCLUSION

Quantum potential barriers have been simulated by a section of waveguide having a particular configuration, and the particle tunneling phenomenon has been simulated by electromagnetic propagation in waveguide. When an effective barrier length of approximately 1.15–1.19 times the actual length is used, there is good correlation with theory in transmission coefficient measurements for both abrupt and slowly varying potential barriers.