Period-speed analysis of a pendulum

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We analyze a simple pendulum by measuring the period and the maximum speed of the bob. Both quantities are measured to high precision using a laser diode and an infrared photodetector located at the bottom of the pendulum. Expressing the period in terms of the maximum speed enables students to examine the large angle dependence of the period, and provides a method to calibrate the speed and do a detailed analysis of the effect of air friction on a sphere. We find that the force due to air friction is well described by a linear and quadratic term in the speed. We investigate the dependence of each term on the sphere's diameter for Reynolds numbers from 250 to 10^4.

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I. INTRODUCTION

Pendula have been studied since the time of Galileo and are a staple in nearly every introductory physics laboratory class. Because they are relatively simple to analyze, they serve as a classic apparatus for demonstrating periodic motion and for measuring the gravitational acceleration g. Many articles[1] have been written on pendulum experiments for the student laboratory, with many of them describing computer-interfaced pendula. A common approach is to collect and analyze data for the angle $\theta(t)$ that the pendulum makes with the vertical as a function of time. These data are often fit by numerical solutions of a differential equation modeling the pendulum, thus requiring an analysis that is mathematically too advanced for most first-year students. In other experiments the period T is measured for different maximum angles $\theta_{\max}$. The drawback of this method is that the angle measurement is not very precise or introduces additional undesirable friction at the pivot point.

In this paper we consider a different set of experimental quantities: The speed at the bottom of the swing, v_i, and the period, T_i, for the ith swing, both of which can be measured precisely with a laser gate located at the bottom of the pendulum. The setup is simple, inexpensive, with little contact friction at the pivot point. The experiment can be analyzed without resorting to the numerical solution of a differential equation, making it well suited for an introductory class. Although measurements of v_i and T_i have been suggested,[2–4] there has been little discussion of how to analyze the motion of the pendulum using just these two variables. We will examine the large angle relation with a simple spreadsheet, and investigate the speed dependence of air friction using the same technique and setup.

II. EXPERIMENTAL SETUP

Our pendulum consists of a spherical bob suspended by two light strings forming a "V" (see Fig. 1), such that the bob stays in the same plane as it swings. Data are taken by a laser gate located at the bottom of the swing. A diode laser shines upon an IR photodetector, which is connected to a personal computer via the printer port. The times that the pendulum blocks the detector are recorded.

This simple and inexpensive laser gate timing setup is used in our first year mechanics class in several experiments. We use a laser diode module found in laser pointers.[5] Most laser diodes designed for laser pointers have an IR mode which is sufficiently strong to be detected by an infrared photodetector. However, not all models feature this IR mode. In comparison to the commonly used infrared LED emitter, the laser pointer has a much smaller diameter. The blocking is therefore much cleaner, and the times can be measured more precisely than with a commercially available infrared LED-detector gate. As an additional benefit, the students can actually see the gate being blocked.

In our setup the infrared photodetector is connected directly across the ground and pin 10 of the parallel port in a personal computer. Pin 10 is the acknowledge (Ack) pin on a standard parallel port. When the diode laser shines on the detector, bit 6 in the status port (baseport +1) registers a 0. When the laser is blocked, the bit reads 1. The program runs in single user mode (init 1) in Linux. The status port is polled by a simple C++ program, and when the bit changes, the real time clock (rtdsc) is read for a time stamp. A similar program runs in DOS.

The data we obtain with this setup are blocking and unblocking times. These times are converted to the maximum speed and period, which we label as v_i and T_i for the ith cycle of the motion. With the strings properly connected, friction at the pivot is negligible, and the main source of dissipation is air friction, which is relatively small. For a starting angle of 40°, for example, the pendulum can swing through more than 300 periods as the angle decreases to 3°.

Because the pendulum is lightly damped, both $\theta_{\max}$ (and therefore the maximum speed $v_{\max}$) and the period T decrease slowly with each half swing. Three consecutive values of $v_{\max}$ are associated with a full period: The speed as the bob starts the cycle, the speed as it swings back through the vertical, and the speed at the end of the cycle. It is very important which one of these three speeds we assign to a given period. We shall show that the best speed to associate with a full cycle is the middle one, half a cycle from the start of the period. For each complete cycle there are six recorded times of blocking and unblocking the photodetector. We calculate the period and corresponding speed as follows. Let b_1 be the time the laser gate is initially blocked and b_2 the time that it becomes unblocked. Halfway through a cycle the bob

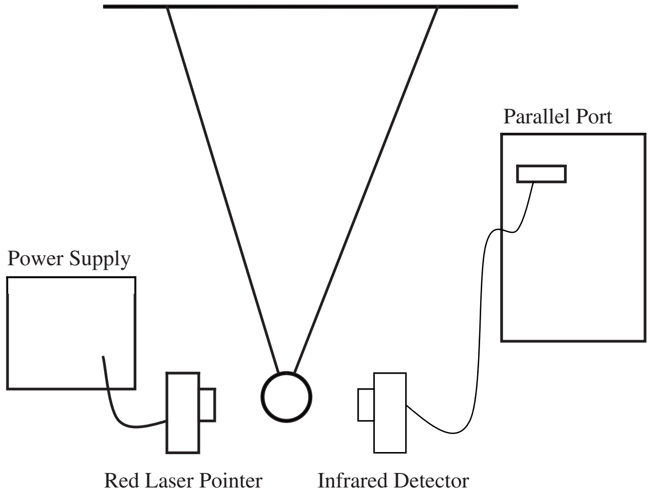

Fig. 1. The setup of the pendulum experiment. A bob is suspended by two light strings. The pendulum bob blocks the laser gate at the bottom of the swing.

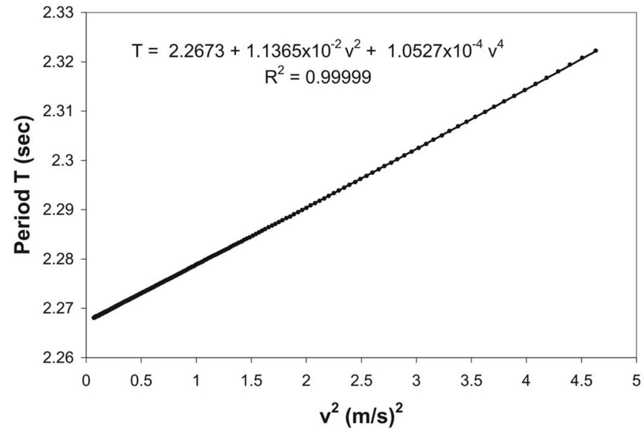

Fig. 2. A plot of the period T_i versus v_i^2, where v_i is the speed at the bottom of the swing for the ith swing of a weakly damped pendulum. The data are fit by a second-order polynomial in v^2 demonstrating the accuracy of Eq. (2).

swings back through the laser gate. Let b_3 be the time the laser is blocked and b_4 the time it becomes unblocked. As the bob swings back completing the cycle, let b_5 be the time the laser gate is blocked and b_6 the time it becomes unblocked. The period is given by $T=(b_5+b_6)/2-(b_1+b_2)/2$. The speed v_i is $w/(b_4-b_3)$, where $w$is the effective thickness of the bob. The maximum speed we record is one half cycle from the start of a period.

Times can be measured to a precision of 2 μs, because the parallel port can be sampled at a frequency above 500 kHz. However, experimental conditions cause the T_i values to deviate more than 2 μs from simple fits to the data. For a stable, well constructed experimental setup the period varies around ± 20 μs from the trend line fit (see Sec. IV). For comparison, commercial infrared IR LED detectors usually have a time measurement uncertainty of ≈ 100 μs. Because the time measurements are so precise, the experimental parameters need to be held very constant. To minimize stretching of the string we clamped the string at the pivot point and used woven instead of nylon string. Air drafts as well as movement in the room need to be eliminated, and the pendulum support must be very stable and not sway as the bob swings. For a consistent blocking distance, the laser should point at the ball's center when it is at the bottom of its swing.

III. EXPERIMENT

The student experiment proceeds as follows. First, the students investigate the dependence of the period T on v_i (or equivalently $\theta_{\max}$). Then, they use their results and the known value for g to calibrate their speed measurements. Once the speed is calibrated, the students determine the force of air friction in terms of the bob's speed. All the analysis is done via a spreadsheet using the $v_i - T_i$ data.

A. Period-speed experiment

After setting up the experiment, the students measure the diameter d of the bob using calipers. This measured diameter will be the approximate distance that blocks the laser gate. The blocking width of the bob w will be slightly less than d due to the finite size of the laser beam. The diameter d is an input parameter for the software to obtain an approximate value for the speed. The students then carefully pull the bob out to an angle of $\approx 40°$ and let it swing. Once the bob is swinging smoothly, the data collection begins. The T_i and v_i data are displayed on the computer screen in real time, and saved for the spreadsheet analysis. Data are collected until $\theta_{\max}$ is around 5°.

The period T of an undamped pendulum depends on the maximum angle $\theta_{\max}$ of the bob. If the experiment is done by lower division students, the expression (1) for T in terms of $\theta_{\max}$ is given to the students. Upper division students see the derivation of $T(\theta_{\max})$, which is found in most junior level mechanics textbooks:[6]

$$T = T_0\left(1 + \frac{1}{4}\sin^2\frac{\theta_{\max}}{2} + \frac{9}{64}\sin^4\frac{\theta_{\max}}{2} + \cdots\right), \tag{1}$$

where $T_0 = 2\pi\sqrt{g'/\ell_{\rm cm}}$, $g' = g\ell_{\rm cm}^2/\ell_G^2$, $\ell_{\rm cm}$ is the distance from the pivot to the center of mass, and ℓ_G is the radius of gyration about the pivot point. For a spherical bob at the end of a massless string $g' = g/(1+(2r^2)/5\ell_{\rm cm}^2)$, where r is the radius of the bob.

Equation (1) for T can be expressed in terms of $v_{\max}$, the speed of the center of mass at the bottom of the swing. If there are no frictional forces, then energy conservation yields $I/(2\ell_{\rm cm}^2)v_{\max}^2 = mg\ell_{\rm cm}(1-\cos\theta_{\max})$, where I is the moment of inertia about the pivot point. Because $\sin^2(\theta/2) = (1-\cos\theta)/2$, we have

$$T = T_0\left(1 + \frac{1}{16}\frac{v_{\max}^2}{g'\ell_{\rm cm}} + \frac{9}{1024}\frac{v_{\max}^4}{g'^2\ell_{\rm cm}^2} + \cdots\right). \tag{2}$$

In Fig. 2 we plot T_i versus v_i^2 for a typical experiment. The bob is a metal sphere of diameter $d = 2.53 \pm 0.01$ cm and mass 66.0 ± 0.1 gm. The length of the pendulum is 128.1 cm. The pendulum started at around 35° and completed over 250 cycles. Because the damping is small, the students analyze the period-speed data neglecting air friction. As seen in Fig. 2, the data lie in an approximate straight line. The straight line demonstrates the significance of the v_i^2 (or $\sin^2\theta_{\max}/2$) dependence on the period T for the range of maximum angles in the experiment. Because every swing is recorded, the analysis is an extensive and accurate verifica-

Table I. The contribution to the period from the first three terms in the expansion of T from Eq. (1). T_0 is taken to be 2 s, which is typical of a classroom pendulum. The times are in units of μs and are listed to an accuracy of two significant figures.

θ_{max}	$(T_0/4)\sin^2(\theta/2)$	$(9T_0/64)\sin^4(\theta/2)$	$(25T_0/256)\sin^6(\theta/2)$
5°	950	1	0.0013
10°	3800	16	0.086
25°	23000	620	20
40°	58000	3800	310

tion of the dependence of T on v_{max}. To obtain the zero angle limit of the period T_0 and to calibrate the speed, the students fit the data from $\theta_{max}=5°$ to 25° with a second-order polynomial in v^2.[7] In Table I we list the contribution for each term in the expansion for an undamped pendulum with period $T_0=2$ s. The level of precision for our setup is around 20 μs (see Fig. 5), so a second-order polynomial fit is sufficient if θ_{max} is less than 25°. The third-order term in the expansion for T is $(25/256)\sin^6\theta/2$.

The students prepare for the air friction experiment by calibrating the speed from the slope of the graph in Fig. 2. If u_i is the measured speed at the bottom of the swing as determined by the laser gate, then $u_i=d/(\text{blocking time})$, where d is the diameter of the bob. The speed u_i will not be the true maximum speed for several reasons: The nonzero thickness of the laser beam, the laser gate might not be located exactly at the center of mass of the bob, and the finite size of the bob. If w is the effective blocking width, the true speed $v_i=w/(\text{blocking time})$. However, the maximum speed at the bottom, v_i, will be proportional to u_i because $v_i/u_i=w/d$ for all speeds. This proportionality begins to fail if the diameter of the bob is a significant fraction of the arc length that the bob travels. For our setup we find that these finite size effects are negligible for angles greater than 3°. If we let $x\equiv v_i/u_i=w/d$, we have

$$T_i \approx 2\pi\sqrt{\frac{\ell_{cm}}{g'}}\left[1+\frac{1}{16}\frac{x^2u_i^2}{g'\ell_{cm}}+\frac{9}{1024}\frac{x^4u_i^4}{g'^2\ell_{cm}^2}+\cdots\right]. \quad (3)$$

The factor of x is a speed calibration factor for the true center of mass maximum speed.

To obtain a calibrated value for v_i we proceed as follows. First we use the data for T_i and u_i for angles between 5° and 25° for which Eq. (2) accurately describes the data. Using the known value for g at the location of the experiment and approximate measurements of ℓ_{cm} and r, we obtain g'. An accurate value for ℓ_{cm} is determined from T_0 and g'. Then x is determined by setting the slope equal to $x^2\sqrt{(\ell_{cm}/g')}\pi/(8g'\ell_{cm})$. Once x is known, the u_i can be correctly scaled to give v_i. This method is the one we used for the speed calibration in the air friction analysis discussed in Sec. III B. We find that $xd\approx d-2$ mm for our laser gate. To test for the accuracy of the calibration, we determined x for different angle ranges of the fit. We find that x varies only by a few percent, and we estimate the error in calibrating v_i to be $\approx 3\%$.

If the experiment is to be used in an introductory laboratory class, the speed calibration factor x can be predetermined and programmed into the software. However, the calibration exercise is desirable, because calibration is an important skill in experimental physics. If a photogate system is purchased commercially, the speed calibration is usually set up by the manufacturer.

B. Air friction experiment

Air friction plays an important role in our daily lives and is of interest to students. A detailed analysis is generally difficult, and the approach taken in introductory mechanics texts is often greatly simplified. Air friction is modeled as a force proportional to the object's speed and/or the square of the speed,[8,9] the former being relevant for laminar flow and the latter for turbulent flow. The v^2 dependence can be tested using coffee filters[10] or balloons, but it is difficult to measure more common objects such as baseballs or golf balls in a laboratory setting, or to examine if there is a linear dependence of the frictional force on the speed. We find that the v_i and T_i pendulum data are well suited for a meaningful treatment of air friction. The analysis can be done using a spreadsheet without requiring a numerical solution of the force equation.

The key quantity to calculate is the following which we define as a_i^*:

$$a_i^* \equiv \frac{v_{i-1}^2-v_{i+1}^2}{2v_iT_i}. \quad (4)$$

We have divided by two because the numerator is the difference of v_i^2 over two cycles. One feature of a_i^* is that the correction for finite bob size (to be discussed later) does not significantly alter the numerator because it is the difference of two squared velocities. Thus, if we neglect the correction due to nonzero bob size, the results will still be fairly accurate. If the magnitude of the force of air friction can be modeled as $f=av+bv^2$, where a and b are constants, then a_i^* is approximately equal to $(av_i+8/(3\pi)bv_i^2)/m$ as can be shown as follows. If we multiply the numerator and denominator in Eq. (4) by the mass m of the bob, we find

$$a_i^* = \frac{2\Delta \mathrm{KE}_i}{mv_iT_i}, \quad (5)$$

where $\Delta\mathrm{KE}_i$ is the change in the kinetic energy in the ith cycle (that is, over one cycle). The kinetic energy loss is equal to the frictional force times the speed integrated over the whole cycle. We have

Table II. The ratio of the actual speed integrals for the pendulum to their values for pure sinusoidal oscillations for different maximum angles θ_{max}. The z_n are defined in the text.

θ_{max}	$z_2=2/(Tv_{max}^2)\int v^2dt$	$z_3=3\pi/(4Tv_{max}^3)\int v^3dt$
10°	0.999	0.999
20°	0.996	0.995
30°	0.991	0.990
40°	0.985	0.982
50°	0.976	0.971
60°	0.964	0.957
90°	0.914	0.899

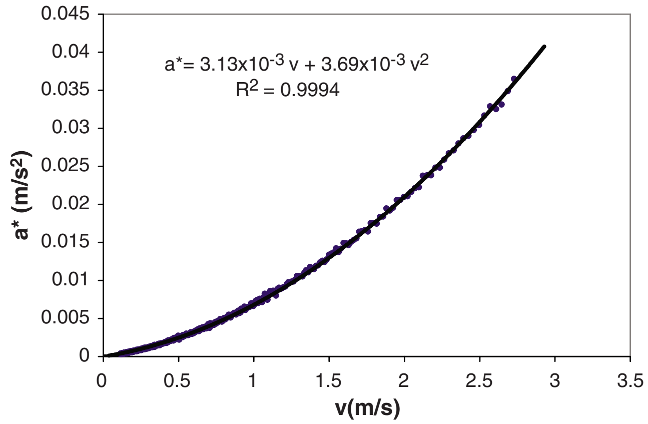

Fig. 3. A plot of a_i^*, defined in Eq. (4), versus v_i for a steel ball of diameter 2.53 cm. The data are well fit by a linear plus quadratic term in v, demonstrating the linear and quadratic dependence of air friction on speed.

$$a_i^* = \frac{2}{mT_iv_i}\int_0^{T_i}(av+bv^2)v\,dt = \frac{2}{mT_iv_i}\int_0^{T_i}(av^2+bv^3)dt. \tag{6}$$

For small oscillation angles the speed of the bob is approximately $v \approx v_i|\cos(2\pi t/T_i)|$. For this sinusoidal approximation for v, we have $\int_0^{T_i} v^2 dt = T_i v_i^2/2$ and $\int_0^{T_i}|v^3|dt = 4T_i v_i^3/(3\pi)$. We substitute these results into Eq. (6) and obtain

$$a_i^* \approx \frac{1}{m}\left[av_i + \left(\frac{8}{3\pi}\right)bv_i^2\right], \tag{7}$$

with the approximation becoming better as $\theta_{\max}$ becomes smaller. To determine the accuracy of the sinusoidal approximation we calculated the v^2 and v^3 integrals numerically for an undamped pendulum. We solved the differential equation numerically for $\theta(t)$ and integrated v^n over a complete cycle. The results are listed in Table II. We define z_n to be the actual (numerical) v^n integral divided by the sinusoidal approximation for v. That is, if $v(t)$ were an exact sinusoidal function, then $z_n=1$. With these corrections, $a_i^*=(1/m)[az_2v_i+8bz_3v_i^2/(3\pi)]$. As seen in Table II, if $\theta_{\max}$ is less than 30°, Eq. (7) is valid to better than 1%. For larger angles the corrections in Table II can be applied.

As a check of the accuracy of Eq. (7) for obtaining a and b from a fit to the data, we solved numerically the equation of motion described in Sec. IV for T_i and v_i. The a_i^* were calculated using this numerical data. Then we fitted the a_i^* versus v_i plot to obtain values for a and b. The values of a and b from the graph of the numerically generated data agreed with the those used in the equation of motion to within 0.1%.

C. Air friction results

In Fig. 3 we plot a_i^* versus v_i for a steel sphere suspended by light strings as in the setup of Fig. 1. The velocities are calibrated using the known value for g as described in Sec. III A. The diameter of the ball is 2.53 cm, and the length of the pendulum $\ell_{\mathrm{cm}}=1.285$ m. A linear plus quadratic form fits the a_i^* very well as seen in Fig. 3.[7] From the fit parameters the students can examine the relative importance of each term. If we define the fit parameters as $a_i^*=c_1v_i+c_2v_i^2$, we have $a=c_1m$ and $b=3\pi c_2 m/8$.

For turbulent flow the force of air friction is proportional to v^2 and given by $f_{\mathrm{turb}}=1/2\,\pi\rho C_d r^2 v^2$ for a sphere,[8] where ρ is the density of the fluid, and C_d is the drag coefficient. We list the b coefficients for five spherical objects of different diameters in Table III along with C_d. For the density of air we choose $\rho_{\mathrm{air}}=1.20$ kg/m^3, its value at room temperature. The Reynolds number for a sphere is given by $R=\rho dv/\eta$, where and $\eta=1.7\times10^{-5}$ Ns/m^2 is the viscosity of air. For Reynolds numbers between 10^3 and 10^5 the drag coefficient for a sphere is around 0.5. As the Reynolds number drops below 10^3, C_d increases. The values in Table III closely follow the graph in Ref.8, Fig. 2. This agreement is remarkable, because the experiments are carried out at speeds below pure turbulent flow. For conditions for which C_d is constant the parameter b should increase as the square of the radius. We graph b as a function of radius squared in Fig. 4(a) to demonstrate the r^2 dependence of the v^2 coefficient.

For laminar flow the air friction force is proportional to v and described by Stokes' law: $f_{\mathrm{lam}}=6\pi\eta rv$ for a sphere.[8] This relation is valid only for Reynolds numbers less than one. For a baseball this condition corresponds to a speed of less than 1 mm/s, much lower than the speeds in our pendulum experiments. Thus, we cannot expect this relation to apply for our data. Still, the plot of a versus r in Fig. 4(b) shows that a is approximately proportional to r, as predicted by Stokes' law. However, the value for η extracted from the linear fit is 73×10^{-5} Ns/m^2 which is around 40 times larger

Table III. Data for several spherical bobs of different diameters. We list the coefficients for the linear and quadratic speed terms. The uncertainties for a and b are due to fitting these parameters to the data. There is an additional uncertainty due to the calibration of v. The range of the Reynolds numbers corresponding to the maximum speed of the bob is also given.

	Steel ball	Golf ball	Rubber ball	Rubber ball	Baseball
Diameter (cm)	2.53	4.26	5.06	5.65	7.2
a ($\times10^{-4}$) N/(m/s)	2.06 ± 0.02	2.44 ± 0.02	3.22 ± 0.03	4.06 ± 0.04	5.18 ± 0.05
b ($\times10^{-4}$) N/(m/s)^2	2.86 ± 0.01	5.53 ± 0.03	6.96 ± 0.04	8.56 ± 0.04	14.30 ± 0.06
Drag coefficient	0.95	0.64	0.57	0.57	0.58
v_{eq} (m/s)	0.72	0.44	0.46	0.47	0.36
Reynolds numbers	270–4000	580–7100	990–9200	1200–9600	1100–9700
$\sqrt{k_4}/k_2$	1.37	1.46	1.45	1.07	1.40

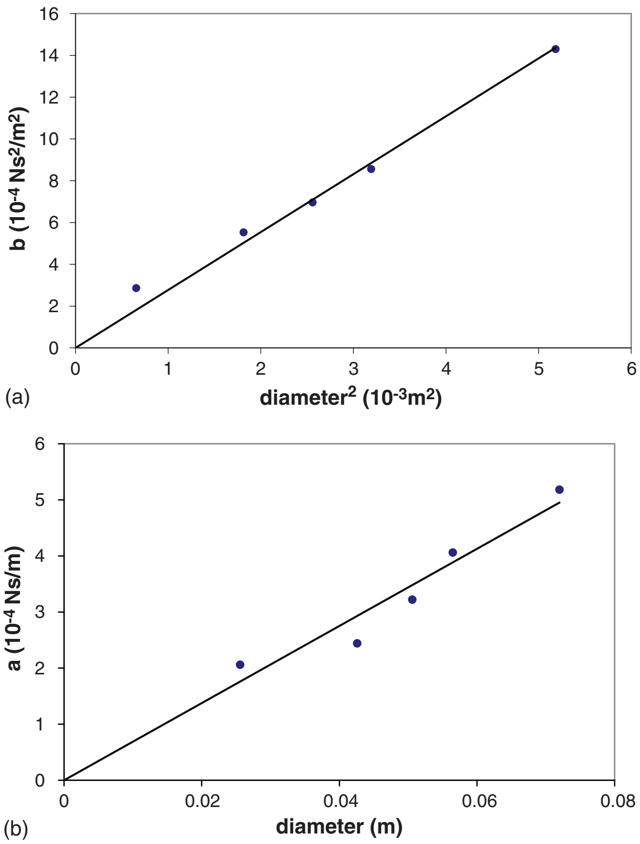

Fig. 4. (a) Plots of the coefficient b of air friction quadratic in v and (b) the coefficient a of air friction linear in v versus the diameter of the five spheres of Table III. The graphs demonstrate the dependence of air friction on the diameter of the bob.

than the accepted value for the viscosity of air. This result is consistent with those of similar experiments.[2,3]

In Table III we list the uncertainty of the coefficients a and b determined from the fit of the data.[11] The uncertainty in a is approximately 1% and that of b around 0.5%. We estimate the overall uncertainty of a to be around 3% and b to be around 6% due to the calibration uncertainty of v_i.

It is beyond the level of an introductory laboratory class to investigate the air friction force for the transition region between laminar and turbulent flow, but this study brings out some interesting points. Even though the speed of the balls is below the turbulent regime, the coefficient of the quadratic term agrees well with the expression for turbulent flow. Because air friction is modeled by both a linear and quadratic speed dependence, it is interesting to determine the speeds at which each of these terms dominate. The speed at which the two terms contribute equally, v_{eq}, is given by the ratio a/b. For a baseball this equality occurs at $v_{eq} \approx 3.5/8.2 = 0.36$ m/s. In Table III we list v_{eq} for the five bobs. In a baseball game the ball usually moves faster than v_{eq}, so the quadratic term is dominant.

IV. ACCURACY OF ANALYSIS

Equation (2) for an undamped pendulum was used to fit the $T_i - v_i$ data. In our setup T and v_{max} can be measured very precisely, so the effects of damping must be considered even though the frictional losses are very small. In practice, all pendula have some damping, so Eq. (2) cannot be used to describe any experiment exactly. In this section we estimate how accurate Eq. (2) describes the data when the effects of damping, truncation errors, and errors involved in measuring v_i are considered.

A. Theory including damping

In the presence of friction v_{max} changes slightly from one swing to the next. Because there are three crossings at $\theta = 0$ for each complete cycle, three values of v_{max} occur during a complete cycle. We associate the period T_i of a cycle with $v_i = v_{mid}$, the speed halfway through a complete cycle. Although this choice is reasonable, we need to examine the validity of Eq. (2) for this choice for a typical student laboratory setup. Using our results from the frictional analysis, we model the damping force by the sum of a linear and quadratic term in the bob's speed. Because we do not know of an analytic solution of the pendulum's motion with this frictional force, we numerically investigate the validity of Eq. (2) with damping included. Specifically, we solve the differential equation $d^2\theta/dt^2 = -g/\ell_{cm}\sin(\theta) - \alpha\ell_{cm}\dot{\theta} - \beta\ell_{cm}^2|\dot{\theta}|\dot{\theta}$ for θ_{max} and T. We choose $g = 9.8$ m/s^2, and $\ell_{cm} = 1.25$ m, which is a typical value. For the frictional acceleration terms, we choose $\alpha = a/m = 0.005$ (m s)$^{-1}$ and $\beta = b/m = 0.005$ m^{-2}, which are roughly the values from our friction analysis. We choose $\theta_{max} = 45°$ initially and compute T_i and $v_i = v_{mid}$ numerically for 230 cycles. After 230 cycles the T_i decreased by around 20 μs, which is typical of the behavior of the experimental data.

To test if Eq. (2) is accurate when v_{mid} is used for v_{max}, we fit T_i versus v_i^2 (that is, v_{mid}^2) for the 230 numerically generated values. Keeping a finite number of terms in Eq. (2) will introduce truncation errors. For angles less than 45° the first term in the expansion is the most significant. For comparison, we fit the numerical data for T_i with a second as well as a sixth order polynomial in v_{mid}^2 for angles between 5° and 25°. A second order polynomial fit gives

$$T = T_0\left(1 + 0.9997\frac{1}{16}\frac{v_{mid}^2}{g'\ell_{cm}} + 1.049\frac{9}{1024}\frac{v_{mid}^4}{g'^2\ell_{cm}^2}\right) \tag{8}$$

and a sixth order fit gives

$$T = T_0\left(1 + 1.0002\frac{1}{16}\frac{v_{mid}^2}{g'\ell_{cm}} + 0.987\frac{9}{1024}\frac{v_{mid}^4}{g'^2\ell_{cm}^2}\right). \tag{9}$$

These results demonstrate that v_{mid} is a good choice for v_i because the corrections to Eq. (2) are very small. The errors due to damping and truncating the series of Eq. (2) do not significantly affect the coefficient of the v_{mid}^2 term. Truncation errors incurred using a second order polynomial are 5% for the v_{mid}^4 coefficient.

The success of using v_{mid} in Eq. (2) can be understood as follows. Consider a full cycle of the pendulum. Let θ_A (θ_B) be the maximum angle of the first (second) half of the cycle. Let T_A (T_B) be the time for the first (second) half of the cycle, and let v_{mid} be the speed of the pendulum as the bob passes back through the origin. Note that $T_A > T_B$ and $\theta_A > \theta_B$. The total period equals $T = T_A + T_B$:

$$T_A \approx \frac{T_0}{2}(1+(1/4)\sin^2(\theta_A/2)+(9/64)\sin^4(\theta_A/2)+\cdots) \tag{10a}$$

$$T_B \approx \frac{T_0}{2}(1+(1/4)\sin^2(\theta_B/2)+(9/64)\sin^4(\theta_B/2)+\cdots). \tag{10b}$$

The speed halfway through a complete cycle, v_{mid}, gives direct information about θ_A and θ_B as can be seen by considering the energy loss during the second and third quarter cycles. The main dissipative force is that of air friction. In general, if the frictional force depends only on $|v|$, then the energy loss in the time τ is $\int_0^\tau |f(v)||v|dt$. Applying the work-energy theorem to the second quarter cycle, we have $mgl(1-\cos(\theta_A))-\int_0^{\tau_A} f(v)\, v dt = mv_{\text{mid}}^2/2$, where τ_A is the time it takes the pendulum to travel from θ_A to the bottom of the swing. We have

$$\sin^2(\theta_A/2) = \frac{v_{\text{mid}}^2}{4\,\ell\, g} + \frac{1}{2mg\,\ell}\int_0^{\tau_A} f(v) v dt. \tag{11}$$

Similarly, if we apply the work-energy theorem to the third quarter cycle, we obtain

$$\sin^2(\theta_B/2) = \frac{v_{\text{mid}}^2}{4\,\ell\, g} - \frac{1}{2mg\,\ell}\int_0^{\tau_B} f(v) v dt. \tag{12}$$

Note that $\tau_A \approx \tau_B \approx T/4$ for light damping, because the half period changes very little from one half cycle to the next. The complete period equals $T_A + T_B$, and the integrals partially cancel in the sum $\sin^2(\theta_A/2)+\sin^2(\theta_B/2)$. This cancellation is not quite as complete for $\sin^4(\theta_B)$ terms. However, because the coefficient of the $\sin^4(\theta_B)$ term is much smaller than that of the $\sin^2(\theta_B/2)$ term, we have:

$$T \approx T_0\left(1 + \frac{1}{16}\frac{v_{\text{mid}}^2}{g'\,\ell_{\text{cm}}} + \cdots\right). \tag{13}$$

From the numerical solutions of Eqs. (8) and (9), which used parameters and conditions applicable to the student laboratory, we see that the v_{mid}^2 coefficient is $1/16$ to within 0.03% including the effects of friction and truncation errors. The numerical solution indicates that friction and truncation effects can modify the coefficient of the v_{mid}^4 term up to 5% of its value for the undamped pendulum.

B. Detailed angular analysis

In Fig. 2 it is seen that the main dependence of the period on $v_{\max}$ is quadratic (that is, it is proportional to $\sin^2\theta_{\max}/2$). For an advanced laboratory exercise the next term in the expansion of T, $v_{\max}^4$ or $\sin^4\theta_{\max}/2$ can be examined. To do this analysis students need to optimize the experimental conditions to eliminate influences that would affect the applicability of Eq. (2). The T versus $v_{\max}^2$ data can be fitted to the polynomial form $T_i = T_0(1+k_2 u_i^2 + k_4 u_i^4)$, where k_2 and k_4 are fitting parameters. Data for angles up to 35° can be included for an accurate determination of k_4. The contribution of the $\sin^4\theta_{\max}/2$ term becomes more significant and the $\sin^6\theta_{\max}/2$ term is still relatively small.

The coefficients k_2 and k_4 are not independent. For an undamped pendulum $\sqrt{k_4}/k_2 = 3/2$. As discussed, damping

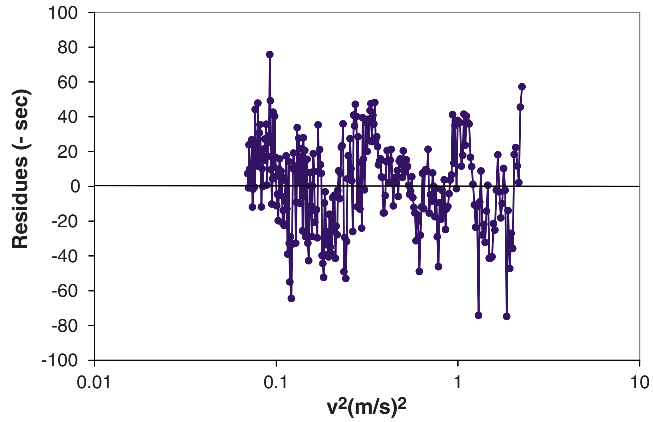

Fig. 5. A plot of the residues from a quadratic fit of the $T_i - v_i^2$ data. A logarithmic scale is used for convenience.

modifies this ratio somewhat by mainly changing k_4 from its undamped value, while the v^2-coefficient k_2 is within 0.4% of the undamped value according to our numerical calculations. In Table III we list this ratio for the spherical bobs we used. For this analysis the fits were for initial angles between 30° and 35° to a final angle of around 5°. Although Eq. (2) describes the data only approximately, the coefficient of u_i^4 is fairly close to the undamped value. This agreement is probably due to the fact that the $\sin^4\theta_{\max}$ term is large enough to produce a measurable effect, as demonstrated in Table I.

C. Measurement errors

In Fig. 5 we plot the residues from a second-order polynomial fit to the data of Fig. 2 for initial angles between 4° and 25°. Because there are more data points for smaller $v_{\max}$ ($v_{\max}$ decreases approximately exponentially), we use a logarithmic scale on the horizontal axis. As seen in Fig. 5, the scatter of the residues is approximately $\pm 20\ \mu$s. For Eq. (2) to accurately describe the data over this range of angles, the conditions of the experiment need to be held constant. For example, if the period of the pendulum is around 2 s, then $\Delta T/T \approx 10^{-5}$. Because $\Delta T/T = (1/2)\Delta\ell/\ell$, $\Delta\ell/\ell$ needs to vary less than 2×10^{-5}. For a pendulum of length 1 m, its length needs to be kept constant to 0.02 mm.

The accuracy of the T versus $v_{\max}$ data gives students an appreciation of the difficulties of doing accurate experiments. Obtaining good fits to T versus $v_{\max}^2$ requires that the experimental parameters remain constant to a few parts in 10^5. Keeping the string length constant, stabilizing the pendulum support, and insuring the laser beam is properly aligned are the most critical features of the apparatus for accurate results. We have tried different setups and found that with reasonable care we can obtain quadratic fits of T versus $v_{\max}^2$ with residues similar to those of Fig. 5 for data taken with $\theta_{\max}$ in the range 25°–5°.

D. Finite size corrections

A systematic error in measuring $v_i = v_{\max}$ is that the speed at the bottom of the swing is not exactly equal to the effective blocking distance divided by the blocking time. This error is because the speed varies during the blocking of the laser gate. Thus, the measured value of $v_{\max}$, v_{meas}, will always be less than $v_{\max}$. If the true $v_{\max}$ equals the same factor times v_{meas} for all $\theta_{\max}$, then the calibration method we have described will correct for this effect. However, this proportionality does not always hold because the fraction of the cycle that the laser gate is blocked depends on the amplitude of the pendulum. The correction is very small, and will be significant only for small angles. The correction is $v_{\max}^2 \approx v_{\text{meas}}^2 + gd^2/(12\ell)$, where v_{meas} is the ratio of the blocking distance to the blocking time.

This relation can be derived by considering the motion of the bob during the time the laser gate is blocked. Because the angle θ the pendulum makes with the vertical is very small during this time, we have to a good approximation $\theta = \omega_m \sqrt{\ell/g} \sin(\sqrt{g/\ell}t)$, where ω_m is the angular speed at the bottom of the swing. If δ is the angle that the pendulum makes just as the gate is blocked and 2τ the total blocking time, we have $\delta = \omega_m \sqrt{\ell/g} \sin(\sqrt{g/\ell}\tau)$. We multiply both sides by ℓ and use $v_{\max} = \ell\omega_m$ and obtain $\tau = \sqrt{\ell/g} \sin^{-1}(\delta\sqrt{\ell g}/v_{\max})$. The expansion of the inverse sin function to leading order in $1/\tau^2$ yields $1/\tau^2 = v_{\max}^2/(\ell\delta)^2(1-\epsilon^2/3-\epsilon^4/15+\cdots)$, where $\epsilon \equiv \delta\sqrt{\ell g}/v_{\max}$. Because $v_{\text{meas}} \approx \ell\delta/\tau$, we have

$$v_{\text{meas}}^2 = v_{\max}^2\left(1 - \frac{\epsilon^2}{3} - \frac{\epsilon^4}{15} + \cdots\right). \tag{14}$$

To leading order in ϵ the result is $v_{\max}^2 \approx v_{\text{meas}}^2 + gd^2/(12\ell)$. That is, the difference between the square of $v_{\max}$ and the square of the measured speed is approximately the same for any speed (or maximum swing angle). This result is reasonable from energy considerations. If we let v_{enter} be the speed of the bob as it just enters the laser gate, we have $v_{\max}^2 - v_{\text{enter}}^2 = 2gh$ where h is the vertical distance that the bob falls from entering the gate to the bottom of the swing. This result is the same for every swing. Thus, it is consistent with energy conservation that the difference in the square of the velocities of $v_{\max}$ and v_{enter} is approximately proportional to h. In our case $h = \ell(1-\cos\delta) \approx \ell\delta^2/2$, and $2gh \approx gd^2/(4\ell)$. The additional factor of $1/3$ in the leading term comes from averaging over the blocking interval. The corrections are small; for the steel ball of Table III, the correction in v is only 0.6% at $\theta_{\max} = 3°$.

V. CONCLUSION

We have shown that it is useful to express the period T of a simple pendulum in terms of $v_{\max}^2$, where $v_{\max}$ is the maximum speed of the center of mass of the pendulum at the bottom of the swing. The advantage of analyzing T as a function of $v_{\max}$ is that $v_{\max}$ and the pendulum's period can be measured very accurately using a laser gate without introducing additional friction at the pivot point. Introductory students can fit the $T_i - v_i^2$ data for maximum angles between 5° and 25° using a second order polynomial. The analysis demonstrates the predominantly $\sin^2\theta_{\max}/2$ (or v_i^2) dependence of the period, gives a reliable extrapolation for T_0, and enables students to do a detailed treatment of air friction. If desired, measurement errors and the contributions of the v_i^4 term in the expansion of T can also be examined.

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[1]We found over 300 articles in this journal and over 150 articles in The Physics Teacher which discussed some aspect of pendula. The ones we have listed include measurements of $v_{\max}$ and/or damping.

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[4]V. K. Gupta, G. Shanker, and N. K. Sharma, "Experiment on fluid drag and viscosity with an oscillating sphere," Am. J. Phys. **54**, 619–622 (1986).

[5]The laser diode we used was purchased from Z-Bolt, ⟨www.z-bolt.com⟩, part number APC-5. We put the laser module in a metal sleeve to facilitate heat transfer.

[6]S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems* (Brooks/Cole, New York, 2004), 5th ed.

[7]We used the spreadsheet Excel because it is commonly used in the student laboratory.

[8]D. Halliday and R. Resnick, *Fundamentals of Physics* (Wiley, New York, 1988), 3rd ed.

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[10]R. A. Serway and J. W. Jewett Jr., *Principles of Physics* (Harcourt Brace Javanovich, San Diego, 2002).

[11]The uncertainty in the parameters of the fit was determined using the graphing program PSI-Plot, ⟨www.polysoftware.com⟩.