Q in mechanical oscillators

A common definition is

2π times the total oscillator's mean stored energy divided by the energy dissipated each cycle.

Q is the figure of merit or quality of an oscillator. inter alia, it is an inverse function of the energy necessary to maintain a given oscillatory amplitude.

NOT ALWAYS!

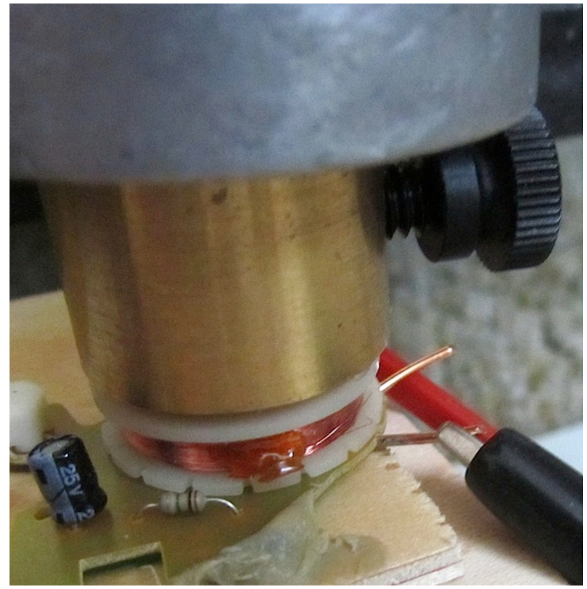
To assume *always* is naive and likely a result of experimental inexperience with mechanical oscillators, or a lack of apprehension of the solutions of the describing differential equations. The result is the assumtion increasing the energy stored (the numerator) is the same as decreasing the dissipation (the denominator) in the Q formulae.

An idea that makes wrong predictions every time is absurd, and is not dangerous, because no one will pay any attention to it. The most dangerous ideas are the those that are often correct or nearly correct, but then betray one at some critical moment.

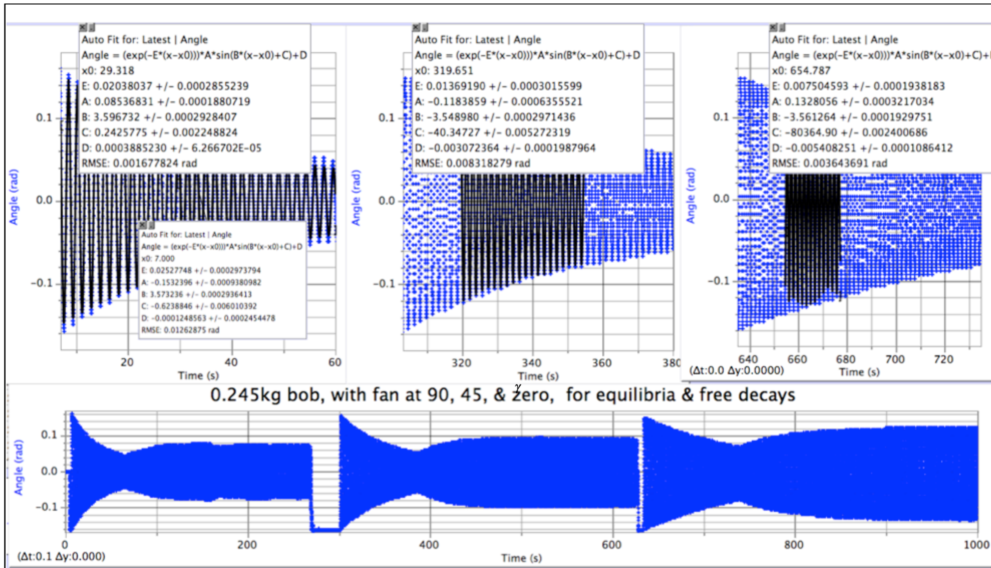
John Denker (paraphrased: Introduction [Ch. 0 of Modern Thermodynamics])

One may easily show that this is not the case for a linearly damped pendulum oscillator experimentally by two driven pendula trials. One, varying the bob mass, while keeping the dissipation approximately constant, and the other for comparison by varying the dissipation with constant mass.

Next is a photograph of a pendulum whose flag's angle changes the dissipation, and the following two graphs, show the resulting equilibria amplitudes and the free decay to determine the Q, for three flag angles.

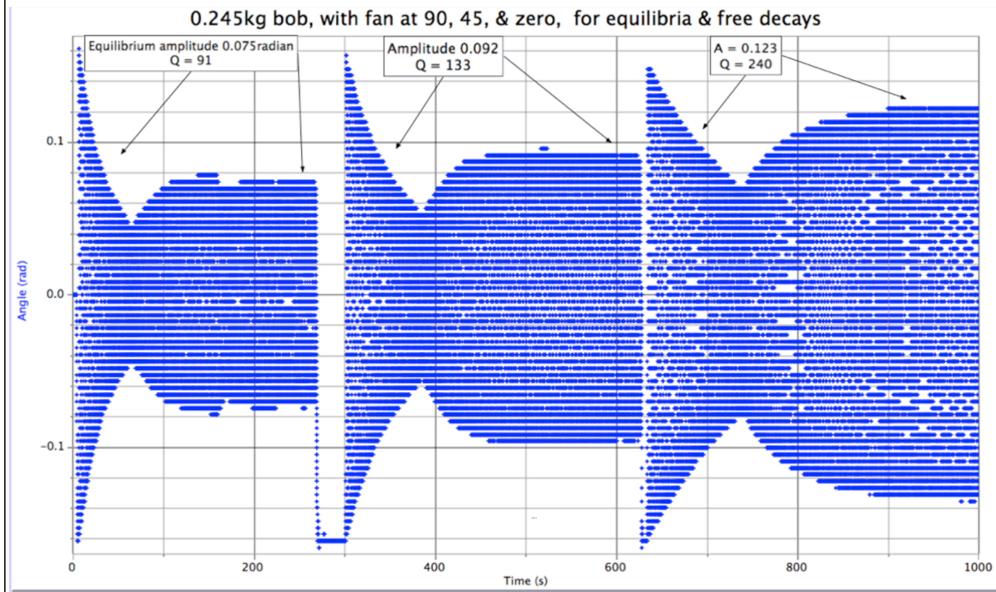


Drive for a faux pendulum from a quartz clock



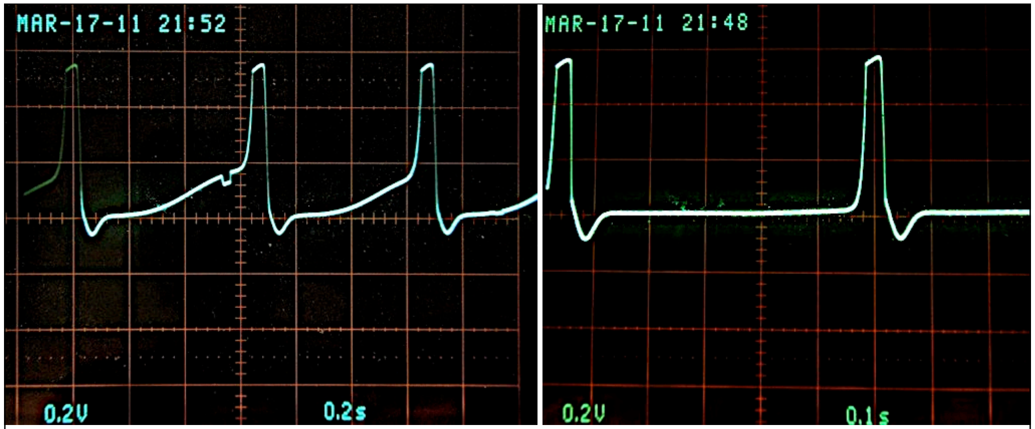
For viscous (linear) dissipation, $Q \simeq \omega_1 / 2(k/m)$ or $\omega_1 m / 2k$

Where ω_1 is the angular frequency and γ is the exponential decay constant, i.e. B and E in the above fits to a linear dissipation model.



Note: The amplitude does increase with Q as expected.

So, as intuitively expected, reduce the drag and the amplitude will increase, as long as the drive is sufficiently constant. This is true enough for my pendula, but as the following oscilloscope pictures show, the drive changes slightly with amplitude.



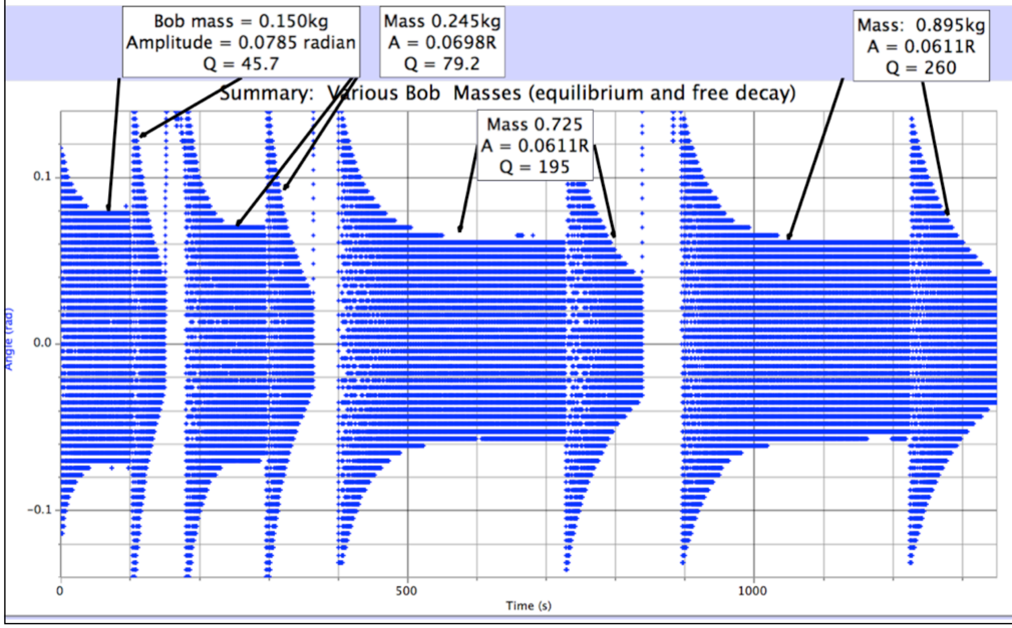
Drive potentials:

Small Amplitude, ~ 0.02R (left) Large Amplitude, > 0.1R (right)

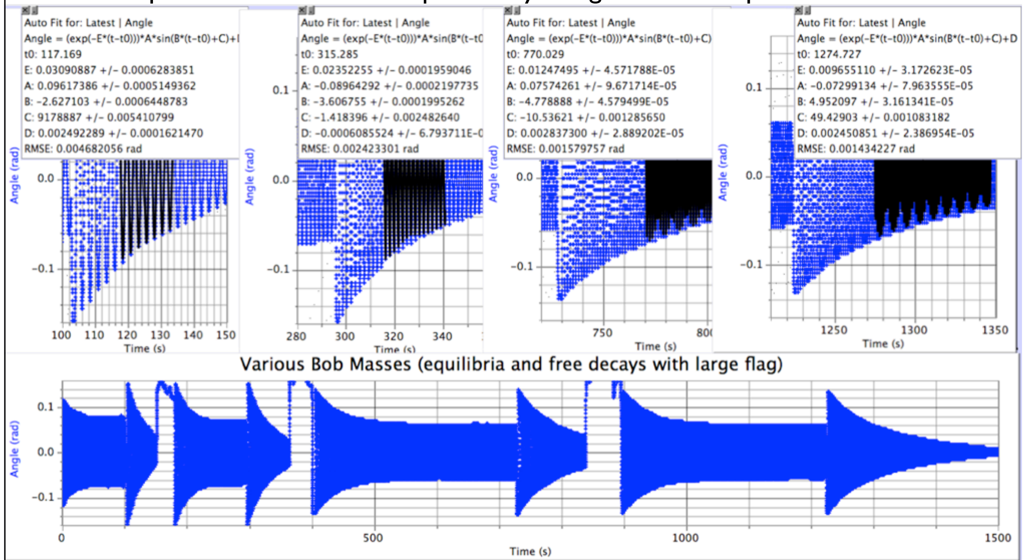
(Note the unfortunate scale change.)

Now the alternate experiment,
varying the mass with approximately constant dissipation.
The same pendulum as pictured earlier was used, but with the flag oriented at maximum drag. This was to reduce all of the Qs to convenient values. With, for example, the maximum mass (0.895kg pictured below) the Q, without the flag, is greater than one thousand. The times to reach equilibrium and to freely decay would be very inconveniently long.

Furthermore, the flag's large dissipation renders the varying drag of the various bob masses negligible. As seen below, the pendulum was stopped after each attainment of equilibrium and free decay, and an additional mass added. Note the amplitude decreased with increasing mass!



Below are the fits to determine the Qs for the various pendula. Note the region for the fits surrounds the amplitude equilibrium values. This was done because the Q values vary with amplitude, which indicates the linear model is only approximate. They are constant in a linear model, and, therefore, there is both turbulent and constant (friction and hysteresis) dissipation in addition to the model's viscous (linear with speed) dissipation. I had found this previously using a three dissipation model fit.



I've shown experimentally, contrary to many's intuition, that the mass of a pendulum's bob does not significantly affect it's driven equilibrium amplitude. However, I had verified in other trials that, characteristically, a driven pendulum responds both in resonance and transient behaviour to its Q, whether the Q value is either due to dissipation and or its bob mass.

So why? Mathematically it is easily seen by keeping ones eyes on the mass. The simplest appropriate model is the differential equation for a harmonically driven linearized pendulum:

$$m\ddot{\theta} + k\dot{\theta} + (mg / l)\theta = \Gamma \cos(\omega t)$$

So both k and gamma (the driving torque) are divided by m.

Therefore, both the Q and the driving amplitude contain m in the denominator.

The particular solution of the above equation for the amplitude at resonance is approximately (very good for large Q):

$$Q(\Gamma / mgl) \text{ or } (\omega_1 m / 2k)\ (\Gamma / mgl)$$

Since $Q \simeq \omega_1 / 2(k / m) \text{ or } \omega_1 m / 2k$

The "m"s cancel!

Finally, and very important, the nature of the driving force is irrelevant to this argument, as long as it is independent of the mass. I used an harmonic drive, because the solution is easily found and illustrated in many texts.

On the other hand, I found, as described widely, a massive pendulum requires a long time to reach equilibrium and has a narrow resonance. Again from the solution: $2\Delta\omega \approx \omega_1 / Q$, where, as before $Q = \omega_1 m / 2k$, therefore, $2\Delta\omega \approx 2k / m$

And the particular solution is:

$$\theta = Be^{-(k/m)t/2} \cos(\omega_1 t + b)$$

So, again, the transient dies slowly if either the bob is massive and or the dissipation weak.

Some of the above mathematical results are intuitive. With increased bob mass the resulting inertia will increase equilibrium time, but adding mass will not reduce the energy to maintain a given amplitude. It does increase the free decay time because the PE is increased proportionally.

Acknowledgements:

I wish to thank B. Mumford of Mumford Micro Systems for insisting pendulum bob mass does not increase pendulum amplitude with constant drive. *Au contraire,* increased bob mass will increase dissipation due to support motion. Mr. Mumford kindly gave me the pendulum drive I used. Also, I used a Vernier Rotary Motion Sensor, and data acquisition system (LabQuest and Logger Pro). The pendulum rod and brass masses are Pasco's, and I cast the lead masses. And, finally, Nancy Seese for giving up our sitting room (pictured next slide), so I may "do" coffee table physics.

Pictured is a pendulum driven electro-magnetically using a rectified sinusoidal EMF of varying frequency.