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Dynamic model of large amplitude non-linear oscillations arising in the structural engineering: Analytical solutions

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ABSTRACT

A mathematical model describing the process of large amplitude vibration of a uniform cantilever beam arising in the structural engineering is proposed. Six different analytical methods are applied to solve the dynamic model of the large amplitude non-linear oscillation equation. Periodic solutions are analytically verified, and consequently the relationship between the natural frequency and the initial amplitude is obtained in an analytical form. Comparison of the present solutions is made with the exact solution and excellent agreement is noted.

1. Introduction

Structural engineering theory is based upon physical laws and empirical knowledge of the structural performance of different landscapes and materials. Many engineering structures can be modelled as a slender, flexible cantilever beam carrying a lumped mass with rotary inertia at an intermediate point along its span; hence they experience large-amplitude vibration [1–5]. In general, such problems are not amenable to exact treatment because of their complexity, and approximate techniques must be resorted to [1–5]. Amongst these, the perturbation methods [6–8] are in common use. Perturbation methods are based on the existence of small parameters, the so-called perturbation quantity.

Recently, considerable attention has been paid towards approximate solutions for analytically solving the nonlinear differential equation. Many nonlinear problems do not contain such a perturbation quantity. So, in order to overcome the shortcomings, many new techniques have appeared in the open literature such as: variational iteration method [9–13], energy balance method [14–20], Hamiltonian approach [21–25], coupled homotopy-variational formulation [26,27], variational approach [28–30], amplitude–frequency formulation [31,32] and other classical methods [33–50].

In this paper, the basic idea of variational approach, energy balance method, Hamiltonian approach, amplitude–frequency formulation and coupled homotopy-variational formulation is introduced and then their applications are studied for the following model of nonlinear oscillations in the engineering structure problems [1]:

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+u+\alpha u^2\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\alpha u\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2+\beta u^3=0 \tag{1}$$

$$u(0)=A,\qquad \frac{\mathrm{d}u}{\mathrm{d}t}(0)=0. \tag{2}$$

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In order to analytically solve this problem, we introduce a new independent variable

$$\tau = \Omega t. \tag{3}$$

Substituting Eq. (3) into Eqs. (1) and (2), we get

$$u'' + u + \alpha u^2 u'' + \alpha u u'^2 + \beta u^3 = 0 \tag{4}$$

with initial conditions:

$$u(0) = A, \qquad u'(0) = 0. \tag{5}$$

The third and fourth terms in Eq. (4) represent inertia-type cubic non-linearity arising from the inextensibility assumption. The last term is a static-type cubic non-linearity associated with the potential energy stored in bending. The modal constants α and β result from the discretization procedure and they have specific values for each mode as described in [1].

2. The application of the variational approach (VA)

The variational approach for nonlinear oscillators was proposed in 2007 [28]. Consider the nonlinear oscillator equation (4). Its variational principle can be obtained by using the semi-inverse method [28]:

$$J(u) = \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{2}\alpha u^2 u'^2 + \frac{1}{4}\beta u^4 \right\} \mathrm{d}t \tag{6}$$

where T is the period of the oscillator $f = \frac{\partial F}{\partial u}$. Assume that its approximate solution can be expressed as:

$$u(t) = A\cos(\omega t). \tag{7}$$

In Eq. (7), ω is the frequency to be determined and A is the amplitude of oscillation. Inserting Eq. (7) into Eq. (6) yields:

$$J = \frac{A^2\pi}{\omega}\left(-\frac{1}{8}\omega^2 - \frac{1}{32}\alpha A^2\omega^2 + \frac{3}{64}\beta A^2 + \frac{1}{8} \right). \tag{8}$$

Using the Ritz method, we obtain:

$$\frac{\partial J}{\partial \omega} = 0, \qquad \frac{\partial J}{\partial A} = 0. \tag{9}$$

In Ref. [28], He gave a very lucid as well as elementary discussion of the invalidity of the Ritz method. In particular, He used an unheard-of simple procedure to arrive at a surprisingly accurate prediction for the relationship between the frequency and amplitude of a nonlinear oscillator. According to Ref. [28], to identify ω one requires

$$\frac{\partial J}{\partial A} = 0, \tag{10}$$

from which the relationship between the amplitude and frequency of the oscillator can be easily obtained:

$$\omega_{VA} = \sqrt{\frac{(3\beta A^2 + 4)}{(2\alpha A^2 + 4)}}. \tag{11}$$

3. The application of the energy balance method (EBM)

In 2002, the energy balance method was proposed by He [18]. In the energy balance method, a variational principle for the nonlinear oscillation is established, then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by the collocation method [28].

In the energy balance method, according to its basic idea, if $\theta = 0$, the whole energy is in the form of kinetic energy and if $\theta = \pi/2$, the whole energy is in the form of potential energy. In $\theta = \pi/4$, there is a balance between the potential energy and kinetic energy [18], from which the angular frequency can be readily obtained by the collocation method. The results are valid not only for weak nonlinear systems, but also for strong nonlinear ones.

It is easy to establish a variational principle for Eq. (4), which reads:

$$J(u) = \int_0^{\frac{T}{4}} \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{2}\alpha u^2 u'^2 + \frac{1}{4}\beta u^4 \right\} \mathrm{d}t, \tag{12}$$

from which its Hamiltonian can be obtained immediately:

$$H(u) = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{2}\alpha u^2 u'^2 + \frac{1}{4}\beta u^4. \tag{13}$$

Or:

$$\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{2}\alpha u^2 u'^2 + \frac{1}{4}\beta u^4 = \frac{1}{2}A^2 + \frac{1}{4}\beta A^4. \tag{14}$$

The simplest trail function is

$$u = A\cos\omega t. \tag{15}$$

Substituting (12) into (11) where $\omega t = \pi/4$, the following residual equation is obtained:

$$R = \frac{1}{4}A^2\omega^2 + \frac{1}{8}\alpha A^4\omega^2 - \frac{1}{4}A^2 - \frac{3}{16}\beta A^4. \tag{16}$$

The first order approximate solution is obtained, which reads:

$$\omega_{EBM} = \sqrt{\frac{(3\beta A^2 + 4)}{(2\alpha A^2 + 4)}}. \tag{17}$$

4. The application of the Hamiltonian approach (HA)

Previously, He [18] had introduced the energy balance method based on collocation and Hamiltonian. Recently, in 2010 it was developed into the Hamiltonian approach [21]. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, the Hamiltonian of Eq. (4) can be written in the form:

$$H(u) = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{2}\alpha u^2 u'^2 + \frac{1}{4}\beta u^4. \tag{18}$$

Eq. (18) implies that the total energy keeps unchanged during the oscillation. According to Eq. (18):

$$\frac{\partial H}{\partial A} = 0. \tag{19}$$

Introducing a new function, $\overline{H}(u)$, defined as [16]:

$$\overline{H}(u) = \int_0^{\frac{T}{4}} \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{2}\alpha u^2 u'^2 + \frac{1}{4}\beta u^4 = \frac{1}{4}TH. \tag{20}$$

It is obvious that:

$$\frac{\partial \overline{H}}{\partial T} = \frac{1}{4}H. \tag{21}$$

Eq. (21) is equivalent to the following one:

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial T}\right) = 0. \tag{22}$$

Or:

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial (1/\omega)}\right) = -\frac{1}{8}\alpha A^3\omega^2\pi - A\omega^2\pi\left(\frac{1}{2} + \frac{1}{8}\alpha A^2\right) + A^3\pi\left(\frac{3}{32}\beta + \frac{1}{16}\alpha\omega^2\right) + A\pi\left(\frac{1}{4}\omega^2 + \frac{1}{16}\alpha A^2\omega^2 + \frac{3}{32}\beta A^2 + \frac{1}{4}\right) = 0. \tag{23}$$

Consequently, the approximate frequency can be found from Eq. (23).

$$\omega_{HA} = \sqrt{\frac{(3\beta A^2 + 4)}{(2\alpha A^2 + 4)}}. \tag{24}$$

5. The application of the integral iteration method (IIM)

In 2005, a simple but effective iteration method was proposed to search for limit cycles or bifurcation curves of nonlinear equations by He [35].

According to the method Eq. (1) can be rewritten in the following iteration form [35]:

$$u_{n+1} = -u_n - f(u_n, u_n', u_n''). \tag{25}$$

Substituting a trail function into (25) and integrating twice, u_{n+1} can be easily obtained.

Consider the nonlinear oscillator equation (4). For $n = 0$, the following iteration form is

$$u_1'' = -u_0 - \alpha u_0^2 u_0'' - \alpha u_0 u_0'^2 - \beta u_0^3. \tag{26}$$

The simplest trail function is

$$u_0 = A\cos\omega t. \tag{27}$$

Substituting Eq. (27) into the functional equation (26) and integrating twice yields:

$$u_1 = \frac{A}{\omega^2}\left(-\frac{1}{2}\alpha A^2\omega^2 + \frac{3}{4}\beta A^2 + A\right)\cos(\omega t) + \frac{A^3}{\omega^2}\left(-\frac{1}{18}\alpha\omega^2 + \beta\right)\cos(3\omega t) + Ct + B \tag{28}$$

C and D are integral constants. The last two terms in (28) do not exhibit periodic behaviour, which is characteristic of oscillator equations, so the terms can be eliminated in the procedure [35].

By equating the coefficients of $\cos(\omega t)$ in u_0 and u_1 the approximate frequency can be obtained:

$$\omega_{IIM} = \sqrt{\frac{(3\beta A^2 + 4)}{(2\alpha A^2 + 4)}}. \tag{29}$$

6. The application of the amplitude–frequency formulation (AFF)

To solve nonlinear problems, an amplitude–frequency formulation for nonlinear oscillators was proposed by He, which was deduced using an ancient Chinese mathematics method [33,34]. According to He's amplitude–frequency formulation, $u_1 = A\cos t$ and $u_2 = A\cos\omega t$ serve as the trial functions. Substituting u_1 and u_2 into Eq. (4) results in the following residuals:

$$R_1 = A^3(\beta - \alpha)\cos^3(t) + \alpha A^3\cos(t)\sin^2(t), \tag{30}$$

and

$$R_2 = \left(A - A\omega^2\right)\cos(\omega t) + A^3\left(\beta - \alpha\omega^2\right)\cos(3\omega t) + \alpha A^3\omega^2\cos(\omega t)\sin^2(\omega t). \tag{31}$$

According to the amplitude–frequency formulation, the above residuals can be rewritten in the forms of weighted residuals:

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\mathrm{d}t, \quad T_1 = 2\pi, \tag{32}$$

and

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\mathrm{d}t, \quad T_2 = \frac{2\pi}{\omega}. \tag{33}$$

Applying He's frequency–amplitude formulation:

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}} \tag{34}$$

where

$$\omega_1 = 1, \qquad \omega_2 = \omega. \tag{35}$$

The approximate frequency can be obtained:

$$\omega_{AFF} = \sqrt{\frac{(3\beta A^2 + 4)}{(2\alpha A^2 + 4)}}. \tag{36}$$

7. The application of the coupled homotopy-variational formulation (CHV)

The coupled method of homotopy perturbation method [38–42] and variational formulation [28–30], couples the homotopy perturbation method with the variational method. The method first constructs a homotopy equation, and then the solution is expanded into a series of p. As the zeroth order approximate solution is easy to be obtained, the second term is solved using the variational approach, where the frequency of the nonlinear oscillator can be obtained. The first-order solution is the best among all possible solutions, when the trial solution is chosen in cosine or sine function. This technology is very much similar to Marinca's work where the unknown parameters are identified using least squares technology [36,37].

The following homotopy can be constructed:

$$u'' + \omega^2 u + p\left[\alpha u^2 u'' + \alpha u u'^2 + \beta u^3 + (1-\omega^2)u\right] = 0, \quad p \in [0, 1]. \tag{37}$$

When $p = 0$, Eq. (37) becomes the linearized equation, $u'' + \omega^2 u = 0$, when $p = 1$, it turns out to be the original one. Assume that the periodic solution to Eq. (4) may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots. \tag{38}$$

Substituting Eq. (38) into Eq. (37) and collecting terms of the same power of p, give:

$$u_0'' + \omega^2 u_0 + = 0, \qquad u_0(0) = A, \qquad u_0'(0) = 0, \tag{39}$$

and

$$u_1'' + \omega^2 u_1 + \alpha u_0^2 u_0'' + \alpha u_0 u_0'^2 + \beta u_0^3 + (1-\omega^2)u_0 = 0, \qquad u_1(0) = 0, \qquad u_1'(0) = 0. \tag{40}$$

The solution of Eq. (39) is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads:

$$J(u_1) = \int_0^T \left\{-\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1-\omega^2)u_0 u_1 + \alpha u_0^2 u_0'' u_1 + \alpha u_0 u_0'^2 u_1 + \beta u_0^3 u_1\right\} \mathrm{d}t, \quad T = \frac{2\pi}{\omega}. \tag{41}$$

To illustrate the procedure better, a simple trail function can be chosen:

$$u_1 = B\left(\cos\omega t - \frac{1}{3}\cos 5\omega t\right). \tag{42}$$

Substituting u_1 into the functional equation (41) results in:

$$J(A, B, \omega) = \frac{\pi B}{\omega}\left(A + \frac{3}{4}\beta A^3 - \frac{1}{2}\alpha A^3\omega^2 - \frac{4}{3}B\omega^2 - A\omega^2\right). \tag{43}$$

Setting:

$$\frac{\partial J}{\partial B} = 0, \qquad \frac{\partial J}{\partial \omega} = 0. \tag{44}$$

Solving the above equations, approximate frequency as a function of amplitude is equals to:

$$\omega_{CHV} = \sqrt{\frac{(3\beta A^2 + 4)}{(2\alpha A^2 + 4)}}. \tag{45}$$

The accuracy of the first-order approximate solution can be dramatically improved if the trail function is chosen:

$$u_1 = B_1\left(\cos\omega t - \frac{1}{3}\cos 3\omega t\right) + B_3\left(\frac{1}{3}\cos 3\omega t - \frac{3}{5}\cos 5\omega t + \frac{5}{7}\cos 7\omega t\right). \tag{46}$$

Substituting Eq. (46) into Eq. (41) leads to the result

$$\begin{aligned} J(A, B_1, B_3, \omega) = {} & \frac{\pi A^3}{\omega}\left(-\frac{1}{6}\alpha B_3\omega^2 - \frac{1}{3}\alpha B_1\omega^2 + \frac{1}{12}\beta B_3 + \frac{2}{3}\beta B_1\right) \\ & + \frac{\pi A}{\omega}\left(B_1 - B_1\omega^2\right) + \pi\omega\left(\frac{8}{9}B_1B_3\omega - \frac{4}{9}B_1^2\omega - \frac{187528}{11025}B_3^2\omega\right). \end{aligned} \tag{47}$$

The stationary condition of Eq. (47) requires that:

$$\frac{\partial J}{\partial B_1} = 0, \qquad \frac{\partial J}{\partial B_3} = 0, \qquad \frac{\partial J}{\partial \omega} = 0. \tag{48}$$

The second-order approximate frequency can be obtained as follows:

$$\begin{aligned}\omega_{CHV2} = {} & \frac{1}{2}\frac{1}{126652\alpha A^2+21517\alpha^2A^4+187528}(\sqrt{2}((126652\alpha A^2+21517\alpha^2A^4 \\ & +187528)(84843\alpha A^4\beta-126652\alpha A^2-375056-250854A^2\beta \\ & +2(7216977924\alpha^2A^8\beta^2+21565644372\alpha^2A^6\beta+127109500932\alpha A^4\beta \\ & +42529125294\alpha A^6\beta^2+16115302204\alpha^2A^4+95003185024\alpha A^2+140667003136 \\ & +188168595648A^2\beta+6294637259)1A^4\beta^2)^{1/2})^{1/2}).\end{aligned} \tag{49}$$

8. Results and discussion

In this section, the applicability, accuracy and effectiveness of the proposed approaches are illustrated by comparing the analytical approximate frequency and periodic solution with the exact solutions [1].

The nonlinear oscillator described in Eq. (4) is a conservative system. By integrating Eq. (4) and using the initial conditions in Eq. (5):

$$\frac{1}{2}\left(1+\alpha u^2\right)u'^2+\frac{1}{2}u^2+\frac{1}{4}\beta u^4=\frac{1}{2}A^2+\frac{1}{4}\beta A^4. \tag{50}$$

From the representation above:

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\pm\left[\frac{2\left(A^2-u^2\right)+\beta\left(A^4-u^4\right)}{2\left(1+\alpha u^2\right)}\right]^{\frac{1}{2}}. \tag{51}$$

The time required for u to change from 0 to A is one-fourth of the exact period $T_{exact}(A)$. Hence:

$$T_{exact}(A)=4\int_0^A\left[\frac{2\left(1+\alpha u^2\right)}{2\left(A^2-u^2\right)+\beta\left(A^4-u^4\right)}\right]^{\frac{1}{2}}\mathrm{d}u. \tag{52}$$

Letting $u=A\cos\omega t$ in Eq. (52) leads to [1]

$$T_{exact}(A)=4\int_0^{\pi/2}\left[\frac{2\left(1+\alpha\left(A\cos\omega t\right)^2\right)}{2\left(A^2-(A\cos\omega t)^2\right)+\beta\left(A^4-(A\cos\omega t)^4\right)}\right]^{\frac{1}{2}}\mathrm{d}\theta. \tag{53}$$

Therefore, the exact frequency is given by:

$$\omega_{exact}(A)=\frac{2\pi}{T_{exact}(A)}. \tag{54}$$

Besides the role of the large amplitude A, a special role is played by the modal constants α and β, which depend on the inertia parameters of the attached inertia element with mass M and rotary inertia J [2]. The simplest cases are when the modal constants α, β are small, because in these cases it is easy to achieve accurate periodic solutions even for large amplitude. Difficulties appear when these modal constants become larger [1,2].

The results obtained for variational approach, energy balance method, Hamiltonian approach, amplitude–frequency formulation, integral iteration method and first-order coupled homotopy-variational formulation are similar and it is also found that, the relative error for any $\alpha>0$, $\beta>0$ is

$$\lim_{A\to\infty}\frac{\omega}{\omega_{exact}}=\frac{2\sqrt{3}}{\pi}\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\cos^2 t}}\mathrm{d}t=0.86603. \tag{55}$$

When $\omega_{VA}=\omega_{EBM}=\omega_{HA}=\omega_{IIM}=\omega_{AFF}=\omega_{CHV}=\omega$.

Also Wu et al. obtained a similar result for this problem by the first order harmonic balance method [1].

For the small modal constants $\alpha=\beta=0.1$, the comparison of the exact frequency ω_{exact}, obtained by integrating Eq. (54), with the first analytical approximate frequencies ω computed using Eqs. (11), (17), (24), (29), (36) and (45) and the first order harmonic balance method is illustrated in Figs. 1–2.

The second-order approximation given by Eq. (49) is actually within 0.7% of the exact frequency for any $\alpha>0$, $\beta>0$ when $A\to\infty$.

$$\lim_{A\to\infty}\frac{\omega_{CHV2}}{\omega_{exact}}==0.9993. \tag{56}$$

Figs. 3–6 indicate that Eq. (49) is more accurate than the first order approximation ω and the second order harmonic balance method [1].

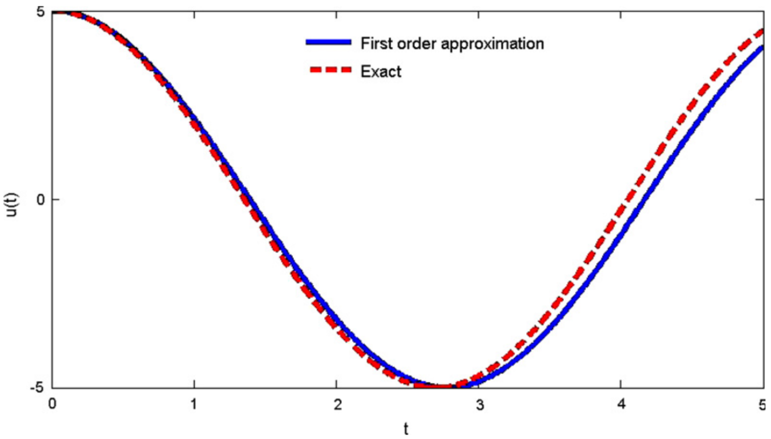

Fig. 1. Comparison of the first order approximate periodic solutions with the exact solution for $\alpha = \beta = 0.1, A = 5$.

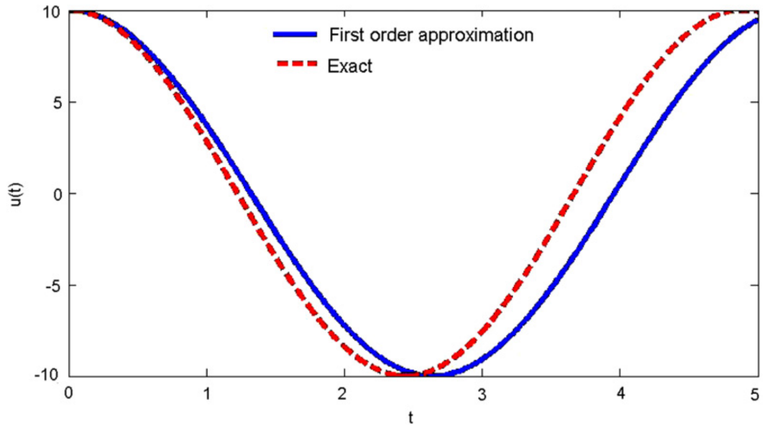

Fig. 2. Comparison of the first order approximate periodic solutions with the exact solution for $\alpha = \beta = 0.1, A = 10$.

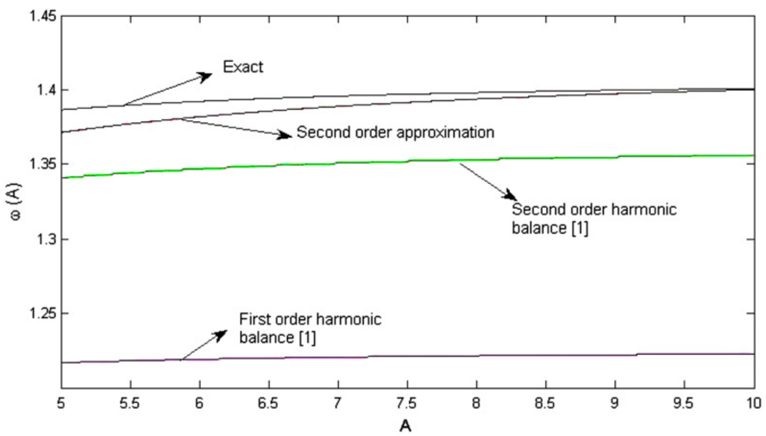

Fig. 3. Dependence of the exact and the analytical approximate frequencies on the amplitude of oscillation for $\alpha = \beta = 1$, for large amplitude.

9. Conclusions

In this paper, six different methods are employed to propose first order and second order approximate solutions for the non-linear oscillation equation arising in nonlinear engineering structures. By introducing these methods for oscillation equations, the following observations have been made:

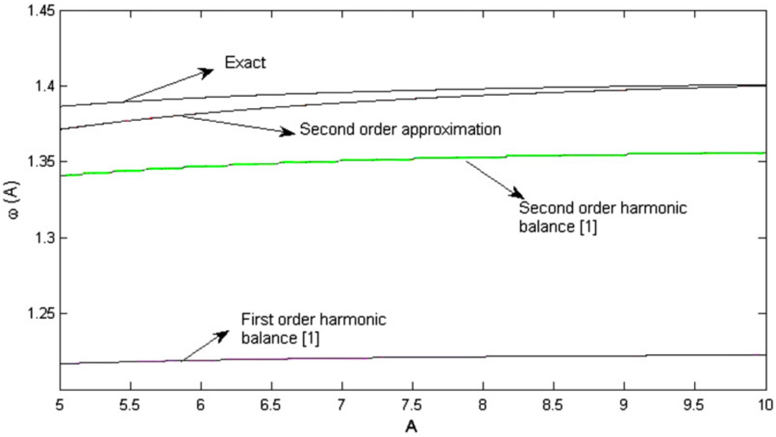

Fig. 4. Dependence of the exact and the analytical approximate frequencies on the amplitude of oscillation for $\alpha = \beta = 2$, for large amplitude.

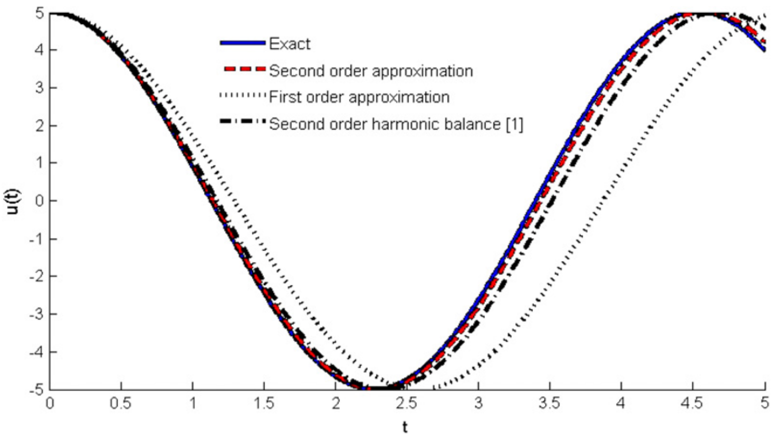

Fig. 5. Comparison of the approximate periodic solutions with the exact solution for $\alpha = \beta = 2, A = 5$.

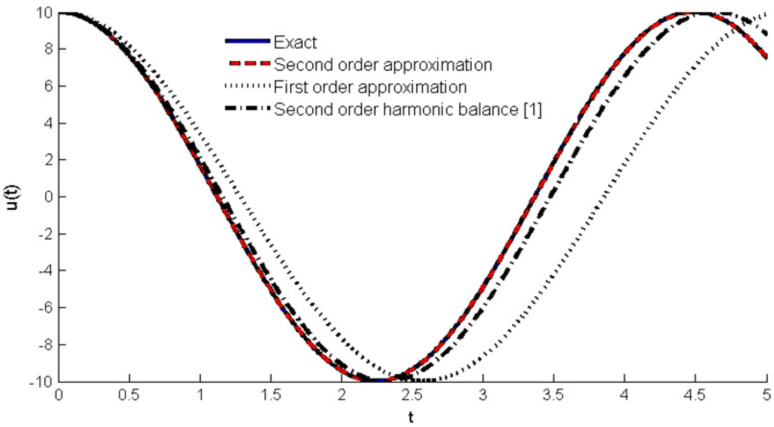

Fig. 6. Comparison of the approximate periodic solutions with the exact solution for $\alpha = \beta = 2, A = 10$.

(1) If the first order approximate solution is required, all the proposed methods can be applied by university students with manual calculation without the requirement of advanced calculus.
(2) The second order approximation obtained with high accuracy by means of the coupled homotopy-variational formulation.

(3) The results obtained for variational approach, energy balance method, Hamiltonian approach, amplitude–frequency formulation, integral iteration method, first order coupled homotopy-variational formulation and first order harmonic balance for the first approximate are similar.
(4) The relative error is smaller for the second order coupled homotopy-variational formulation than the second order harmonic balance.
(5) In coupled homotopy-variational formulation, the third- or higher-order approximates can readily obtain with high accuracy.
(6) It is obvious that the variational approach provides us with a freedom of choice of trial function and gives us more information on the relation between frequency and amplitude.

Convergence and error study for the above mentioned methods are further needed and it is clear that many other modifications can be made.

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